Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Solutions to Assignment #7
The Linear Order on $\mathbb{Z}$
Due on Friday, 31 October.

Recall that a (strict) linear order on a set A, let's denote it by $\triangleleft$, is a binary relation satisfying the following conditions:

1. *Irreflexivity:* For all $a \in A$, it is not the case that $a \triangleleft a$.
2. *Transitivity:* For all a, b, $c \in A$, if $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.
3. *Trichotomy:* For all a, $b \in A$, exactly one of $a \triangleleft b$, $a = b$, or $b \triangleleft a$, is true.

Recall also that we defined the integers to be the set of equivalence classes of the equivalence relation $\sim$ on $\mathbb{N} \times \mathbb{N} = \{\, (a,b) \mid a,\, b \in \mathbb{N} \,\}$ given by $(a,b) \sim (c,d) \iff a + d = b + c$. The equivalence class of (a,b) is then $[(a,b)]_\sim = \{\, (c,d) \in \mathbb{N} \times \mathbb{N} \mid (a,b) \sim (c,d) \,\}$ and the set of integers is $\mathbb{Z} = \{\, [(a,b)]_\sim \mid a,\, b \in \mathbb{N} \,\}$.

We can define the usual linear order on the integers in several ways. Your task, should you choose to accept it, is to ...

1. Give a formal definition of the linear order, let's call it $<_\mathbb{Z}$, on the integers. *[5]*

SOLUTION. We define $<_\mathbb{Z}$ as follows:

$$[(a,b)]_\sim <_\mathbb{Z} [(c,d)]_\sim \iff a + d <_\mathbb{N} b + c$$

Intuitively, the equivalence classes $[(a,b)]_\sim$ and $[(c,d)]_\sim$ represent the differences $a - b$ and $c - d$, so $a - b < c - d \iff a + d < b + c$.

We ought to check that $<_\mathbb{Z}$ is "well-defined", that is, the relation does not depend on the particular choice of representatives from each equivalence class. This means that given that $[(a,b)]_\sim = [(a',b')]_\sim$ and $[(c,d)]_\sim = [(c',d')]_\sim$, we need to check that

$$[(a,b)]_\sim <_\mathbb{Z} [(c,d)]_\sim \iff [(a',b')]_\sim <_\mathbb{Z} [(c',d')]_\sim \,.$$

Note that for the equivalence classes to be equal, we must have that $(a,b) \sim (a',b')$, *i.e.* $a + b' = b + a'$, and $(c,d) \sim (c',d')$, *i.e.* $c + d' = d + c'$. Here we go, implicitly using various facts about addition and the linear order on $\mathbb{N}$:

$$\begin{aligned}
[(a,b)]_\sim <_\mathbb{Z} [(c,d)]_\sim &\iff a + d = b + c \\
&\iff a + d + a' + b' + c' + d' <_\mathbb{N} b + c + a' + b' + c' + d' \\
&\iff (a' + d') + (a + b') + (c' + d) <_\mathbb{N} (b' + c') + (b + a') + (c + d') \\
&\iff a' + d' <_\mathbb{N} b' + c' \\
&\iff [(a',b')]_\sim <_\mathbb{Z} [(c',d')]_\sim
\end{aligned}$$

Thus $<_\mathbb{Z}$ is well-defined. ■

2. Show that $<_{\mathbb{Z}}$ is indeed a linear order on $\mathbb{Z}$. You may assume that we know everything you might need to know about the natural numbers to execute your proof. *[5]*

SOLUTION. We check that the three conditions for being a linear order are true of $<_{\mathbb{Z}}$.

1. $<_{\mathbb{Z}}$ *is irreflexive.* $[(a,b)]_\sim <_{\mathbb{Z}} [(a,b)]_\sim$ would require that $a+b <_{\mathbb{N}} b+a$, which is not true because $a+b = b+a$ and $<_{\mathbb{N}}$ is irreflexive. Thus $<_{\mathbb{Z}}$ is irreflexive.

2. $<_{\mathbb{Z}}$ *is transitive.* Suppose $[(a,b)]_\sim <_{\mathbb{Z}} [(c,d)]_\sim$ and $[(c,d)]_\sim <_{\mathbb{Z}} [(e,f)]_\sim$, so $a+d <_{\mathbb{N}} b+c$ and $c+f <_{\mathbb{N}} d+e$. It follows that, using various properties of $+$ and $<_{\mathbb{N}}$ on $\mathbb{N}$,

$$\begin{aligned}(a+d)+(c+f) <_{\mathbb{N}} (b+c)+(d+e) &\implies (a+f)+(d+c) <_{\mathbb{N}} (b+e)+(c+d)\\ &\implies a+f <_{\mathbb{N}} b+e,\end{aligned}$$

and so, by definition, $[(a,b)]_\sim <_{\mathbb{Z}} [(e,f)]_\sim$. Thus $<_{\mathbb{Z}}$ is transitive.

3. $<_{\mathbb{Z}}$ *satisfies trichotomy.* Consider $[(a,b)]_\sim, [(c,d)]_\sim \in \mathbb{Z}$. Since $<_{\mathbb{N}}$ is a linear order, it satisfies trichotomy, so exactly one of $a+d <_{\mathbb{N}} b+c$, $a+d = b+c$, or $b+c <_{\mathbb{N}} a+d$ is true. By the definition of $<_{\mathbb{Z}}$, it follows that exactly one of $[(a,b)]_\sim <_{\mathbb{Z}} [(c,d)]_\sim$, $[(a,b)]_\sim = [(c,d)]_\sim$, or $[(c,d)]_\sim <_{\mathbb{Z}} [(a,b)]_\sim$ is true. Thus $<_{\mathbb{Z}}$ satisfies trichotomy. ■