Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Assignment #6
The Linear Order on $\mathbb{N}$
Due on Friday, 17 October.[*]

As was noted in class some time ago, a (strict) linear order on a set A, let's denote it by $\triangleleft$, is a binary relation satisfying the following conditions:

1. *Irreflexivity:* For all $a \in A$, it is not the case that $a \triangleleft a$.
2. *Transitivity:* For all a, b, $c \in A$, if $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.
3. *Trichotomy:* For all a, $b \in A$, exactly one of $a \triangleleft b$, $a = b$, or $b \triangleleft a$, is true.

We can define the usual linear order on the natural numbers is several ways. Here is perhaps the most common one:

DEFINITION. $a < b$ for natural numbers a and b if and only if $b = a + S(k)$ for some $k \in \mathbb{N}$.

This definition plays nicely with our construction of the natural numbers in set theory:

$$\begin{aligned}
&0 = \emptyset \\
&1 = S(0) = \{0\} \\
&2 = S(1) = \{0, 1\} \\
&3 = S(2) = \{0, 1, 2\} \\
&\quad\vdots \\
&n + 1 = S(n) = \{0, 1, 2, \dots, n\} \\
&\quad\vdots\,,
\end{aligned}$$

where the successor function is defined by $S(x) = x \cup \{x\}$ for any set x. In particular, the following equivalence is left to you to prove.

1. Show that for all a, $b \in \mathbb{N}$, $a < b$ if and only if $a \in b$. *[5]*

2. Show that $<$ on $\mathbb{N}$ is a linear order. *[5]*

[*] Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can,