Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Solutions to Assignment #6
The Linear Order on $\mathbb{N}$
Due on Friday, 17 October.

As was noted in class some time ago, a (strict) linear order on a set A, let's denote it by $\lhd$, is a binary relation satisfying the following conditions:

1. *Irreflexivity:* For all $a \in A$, it is not the case that $a \lhd a$.
2. *Transitivity:* For all a, b, $c \in A$, if $a \lhd b$ and $b \lhd c$, then $a \lhd c$.
3. *Trichotomy:* For all a, $b \in A$, exactly one of $a \lhd b$, $a = b$, or $b \lhd a$, is true.

We can define the usual linear order on the natural numbers is several ways. Here is perhaps the most common one:

DEFINITION. $a < b$ for natural numbers a and b if and only if $b = a + S(k)$ for some $k \in \mathbb{N}$.

This definition plays nicely with our construction of the natural numbers in set theory:

$$\begin{aligned}
&0 = \emptyset \\
&1 = S(0) = \{0\} \\
&2 = S(1) = \{0, 1\} \\
&3 = S(2) = \{0, 1, 2\} \\
&\quad\vdots \\
&n + 1 = S(n) = \{0, 1, 2, \ldots, n\} \\
&\quad\vdots\,,
\end{aligned}$$

where the successor function is defined by $S(x) = x \cup \{x\}$ for any set x. In particular, the following equivalence is left to you to prove.

1. Show that for all a, $b \in \mathbb{N}$, $a < b$ if and only if $a \in b$. *[5]*

SOLUTION. We will proceed by induction on b.

Base Step. ($b = 0$) Observe that if $b = 0 = \emptyset$, then $b \neq S(a + k) = a + S(k)$, *i.e.* $a \not< b$, for every $a \in \mathbb{N}$ simply because $b = 0$ is the only natural number which is not a successor, and also $a \notin \emptyset = b$ for every $a \in \mathbb{N}$. Thus $a < b$ if and only if $a \in b$ when $b = 0$, for all $a \in \mathbb{N}$.

Induction Hypothesis. ($b = n$) $a < n$ if and only if $a \in n$ for all $a \in \mathbb{N}$.

Induction Step. ($b = n \to b = S(n)$) We need to show that $a < b = S(n) \iff a \in b = S(n)$.

[$\Longrightarrow$] Suppose $a < b = S(n)$, *i.e.* $b = S(n) = a + S(k)$ for some integer k. Then $S(n) = a + S(k) = S(a + k)$, so $n = a + k$ because S is 1–1. If $k = 0$, then $a = a + 0 = n \in n \cup \{n\} = S(n) = b$. On the other hand, if $k \neq 0$, then $k = S(m)$ for some integer m. In this case, $n = a + k = a + S(m)$, *i.e.* $a < n$, so $a \in n$ by the induction hypothesis, and it follows that $a \in n \cup \{n\} = S(n) = b$.

[$\Longleftarrow$] Suppose $a \in b = S(n) = n \cup \{n\}$. Then either $a = n$ or $a \in n$. If $a = n$, then $b = S(n) = S(n+0) = n + S(0) = a + S(0)$, so $a < b$. On the other hand, if $a \in n$, then $a < n$ by the induction hypothesis, *i.e.* $n = a + S(k)$ for some integer k, so $b = S(n) = S\left(a + S(K)\right) = a + S\left(S(k)\right)$, *i.e.* $a < b$.

Thus, by mathematical induction, $a < b$ if and only if $a \in b$ for all a, $b \in \mathbb{N}$. ∎

2. Show that $<$ on $\mathbb{N}$ is a linear order. *[5]*

SOLUTION. We check the three conditions that $<$ must satisfy to be a linear order.

1. We know that one of the consequences of the Axiom of Foundation is that $a \notin a$ for all sets a, and hence for all $a \in \mathbb{N}$. By question **1**, it follows that $a \not< a$ for all $a \in \mathbb{N}$, so $<$ is irreflexive.

2. Suppose $a < b$ and $b < c$ for some a, b, $c \in \mathbb{N}$. By definition, this means that $b = a + S(k)$ and $c = b + S(m)$ for some natural numbers k and m. Then $c = b + S(m) = (a + S(k)) + S(m) = a + (S(k) + S(m)) = a + S\left(S(k) + m\right)$, so $a < c$ by definition. Thus $<$ is transitive.

3. We will use induction on $b \in \mathbb{N}$ to show that for all $a \in \mathbb{N}$, exactly one of $a < b$, $a = b$, or $b < a$ must be true.

Base Step. ($b = 0$) Since $b = 0 = \emptyset$, we cannot have $a \in b$, so $a \not< b$ by **1**. If $a = 0$, then $a = b$, and we cannot also have $b < a$ because this would mean that $b \in a = 0 = \emptyset$ by **1**. If $a \neq 0$, then $a \neq b = 0$ immediately, and $a = S(k)$ for some $k \in \mathbb{N}$, so $a = 0 + a = 0 + S(k) = b + S(k)$, which means that $b < a$ by definition.

Induction Hypothesis. ($b = n$) For all $a \in \mathbb{N}$, exactly one of $a < n$, $a = n$, or $n < a$ is true.

Induction Step. ($b = n \to b = S(n)$) We need to show that exactly one of $a < S(n)$, $a = S(n)$, or $S(n) < a$ is true. By the induction hypothesis, exactly one of $a < n$, $a = n$, or $n < a$ is true. There are three cases:

Case 1. If $a = n$, then $a + S(0) = S(a + 0) = S(a) = S(n)$, so $a < S(n)$ by definition.
Case 2. If $a < n$, then $n = a + S(k)$ for some k, but then $S(n) = S\left(a + S(k)\right) = a + S\left(S(k)\right)$, so $a < S(n)$ by definition.
Case 3. If $n < a$, then $a = n + S(k)$ for some k by definition. In the subcase that $k = 0$, we have that $a = n + S(0) = S(n + 0) = S(n)$. In the subcase that $k \neq 0$, we must have that $k = S(m)$ for some m. Then $a = n + S(k) = n + S\left(S(k)\right) = S\left(n + S(k)\right) = S\left(S(k) + n\right) = S(k) + S(n) = S(n) + S(k)$, so $S(n) < a$ by definition.

It follows that exactly one of $a < S(n)$, $a = S(n)$, or $S(n) < a$ is true.

Thus, by mathematical induction, exactly one of $a < b$, $a = b$, or $b < a$ must be true for all a, $b \in \mathbb{N}$, *i.e.* $<$ satisfies trichotomy.

Since it satisfies all of the necessary conditions, $<$ is a linear order on $\mathbb{N}$. ■