Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Solutions to Assignment #4
A little pidgeon quibbled with my Aristotelian logic ...
Due on Friday, 3 October.

Suppose we have a first-order language whose symbols are the necessary ones (variables, connectives, quantifiers, and equality), plus two one-place relations, P and Q, with the usual formation rules for formulas. Px is supposed to mean "x is a pidgeon" and Qx is supposed to mean "x is a quibbler".

1. Translate the four sentences

All pidgeons are quibblers.
Some pidgeons are quibblers.
No pidgeons are quibblers.
Some pidgeons are not quibblers.

into logically equivalent formulas of the first-order language described above. *[4]*

NOTE: The four sentences reflect the four main sentence forms involving quantifiers studied in Aristotle's logic.

SOLUTION. Here we go!

All pidgeons are quibblers. Translates as $\forall x(Px \to Qx)$.
Some pidgeons are quibblers. Translates as $\exists x(Px \land Qx)$.
No pidgeons are quibblers. Translates as $\forall x(Px \to (\neg Qx))$, or $\forall x(Qx \to (\neg Px))$, or $(\neg \exists x(Px \land Qx))$. [Why are these equivalent?]
Some pidgeons are not quibblers. Translates as $\exists x(Px \land (\neg Qx))$.

And so, there we are! ■

2. If there are no pidgeons, but there are some quibblers, in the universe the four sentences in question **1** are talking about, which of them must be true? If, instead, there are no quibblers, but there are some pidgeons in that universe, which of the four sentences must be true? (Explain why in each case.) *[3]*

SOLUTION. First, suppose there are no pidgeons in the "universe of discourse," but there are some quibblers.

- Since there are no pidgeons, Px is false for all x, so $Px \to Qx$ is true for all x, *i.e.* $\forall x(Px \to Qx)$ is true.
- Since there are no pidgeons, Px is false for all x, so $Px \land Qx$ is false for all x, *i.e.* $\exists x(Px \land Qx)$ must be false.
- Since there are no pidgeons, Px is false for all x, so $Px \to (\neg Qx)$ is true for all x, *i.e.* $\forall x(Px \to (\neg Qx))$ is true.
- Since there are no pidgeons, Px is false for all x, so $Px \land (\land Qx)$ is false for all x, *i.e.* $\exists x(Px \land (\neg Qx))$ must be false.

Second, suppose there are no quibblers, but there are some pidgeons in the universe of discourse.

- Since there are no quibblers, but there are some pidgeons, Px is true for some x for which Qx is false, so $Px \to Qx$ is false for some x, *i.e.* $\forall x(Px \to Qx)$ is false.

- Since there are no quibblers, Qx is false for all x, so $Px \wedge Qx$ is false for all x, *i.e.* $\exists x(Px \wedge Qx)$ must be false.
- Since there are no quibblers, Qx is false for all x, so $\neg Qx$ is true for all x, so $Px \to (\neg Qx)$ is true for all x, *i.e.* $\forall x(Px \to (\neg Qx))$ is true.
- Since there are no quibblers but there are some pidgeons, Px is true for some x for which Qx is false and hence $\neg Qx$ is true, so $Px \wedge (\neg Qx)$ is true for some x, *i.e.* $\exists x(Px \wedge (\neg Qx))$ must be true.

  And that's all, folks! ■

3. Aristotelian logic pretty much consists of propositional logic, plus just enough first-order logic to properly handle the four sentence forms given above. Give an example of a mathematical result and its proof that Aristotelian logic is *not* adequate to handle, and explain why this is so. *[3]*

Hint. Look for a definition with alternating universal and existential quantifiers and then a result that depends on that definition.

SOLUTION. Consider the ε–δ definition of limits:

$$\lim_{x \to a} f(x) = L \iff \begin{array}{l} \text{For all } \varepsilon > 0, \text{ there is a } \delta > 0, \text{ such that} \\ \text{for all } x, \text{ if } |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon. \end{array}$$

This has three quantifiers, with an existential quantifier sandwiched between two universal quantifiers. It's hard to see how being able to handle the only the four formula schema give in **1** could allow one to disentangle the ε–δ definition of limits in an application of the definition, such as verifying the Sum Rule for limits. ■