Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2025

Assignment #3
Lengths
Due on Friday, 26 September.

Please read the handout *A Minimal System of Propositional Logic* before tackling this assignment.

1. Consider the formulas of our official system of propositional logic, $\mathcal{L}_P$, that do *not* have any instance of the connective $\rightarrow$. Determine, with proof, exactly what the possible lengths of such formulas are. *[10]*

Note. The length of a formula is its length as a string: the number of symbols of the language it contains, counting repetitions.

Solution. A formula of our official system of propositional logic, $\mathcal{L}_P$, that does not have any instance of the connective $\rightarrow$ is either atomic or built from an atomic formula with some finite number of applications of negation. We claim that the length of such a formula must be $3n+1$, where $n \geq 0$ is the number of applications of negation used to build the formula, and we prove this by induction on n.

Base Step. $(n=0)$ If the number of applications of negation used to construct a formula is $n=0$, and the formula does not use the connective $\rightarrow$, then the formula has no connectives. The only formulas of $\mathcal{L}_P$ without connectives are the atomic formulas, which have length $1 = 3 \cdot 0 + 1 = 3n+1$ as strings of symbols of the language.

Inductive Hypothesis. $(n=k)$ If a formula α of $\mathcal{L}_P$ has no instance of $\rightarrow$ and was built using $k \geq 0$ applications of negation, then α has length $3k+1$.

Inductive Step. $(n=k \Rightarrow n=k+1)$ Suppose β is a formula which has no instance of $\rightarrow$ and was built using $n=k+1$ applications of negations. Then β must be $(\neg\alpha)$ for some formula α which has no instance of $\rightarrow$ and was built using k applications of negation. By the inductive hypothesis, α must have length $3k+1$ and, since β has the three additional symbols (, $\neg$, and), it follows that β has length $3k+1+3 = 3(k+1)+1 = 3n+1$, as required.

By mathematical induction, it follows that a formula of our official system of propositional logic, $\mathcal{L}_P$, that does not have any instance of the connective $\rightarrow$ has length $3n+1$, where n is the number of applications of negation used to build the formula. ■

Note 1. It should be pretty obvious, too, that if $n \geq 0$ and we want to build a formula of $\mathcal{L}_P$ of length $3n+1$ and with no instance of $\rightarrow$, we can do so by starting with an atomic formula and negating it n times.

Note 2. In case you're paranoid, why does the above proof guarantee that there is no formula which has no instance of $\rightarrow$ and which has a length that is not of the form $3n+1$ for some $n \geq 0$?