Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Solutions to Assignment #11
Real Addition
Due on Friday, 28 November.

Recall that we defined the real numbers to be the *schnitts* or *Dedekind cuts*. A schnitt is a subset $r \subseteq \mathbb{Q}$ satisfying

i. $r \neq \emptyset$ and $r \neq \mathbb{Q}$;
ii. if $p \in r$ and $q < p$ for some $q \in \mathbb{Q}$, then $q \in r$ (r is "downward closed");
iii. if $p \in r$, then there is some $q \in r$ with $p < q$ (r has no largest element).

Officially, $\mathbb{R} = \{ r \subset \mathbb{Q} \mid r \text{ is a schnitt} \}$. We then proceeded to define addition on $\mathbb{R}$ by:

$$r +_{\mathbb{R}} s = \left\{ p +_{\mathbb{Q}} q \mid p \in r \text{ and } q \in s \right\}$$

We verified in class that this is a schnitt, too.

1. Verify that the addition of real numbers (*i.e.* schnitts) is associative and commutative. *[5]*

NOTE. You may assume that the rational numbers and addition of rational numbers have all the properties you need.

SOLUTION. We'll do commutativity first. Suppose r and s are any schnitts. Then

$$\begin{aligned} r +_{\mathbb{R}} s &= \left\{ p +_{\mathbb{Q}} q \mid p \in r \text{ and } q \in s \right\} \\ &= \left\{ q +_{\mathbb{Q}} p \mid q \in s \text{ and } p \in r \right\} \quad \text{[By the commutativity of } +_{\mathbb{Q}}.] \\ &= s +_{\mathbb{R}} r, \end{aligned}$$

so $+_{\mathbb{R}}$ is commutative.

Now for associativity. Suppose r, s, and t are any schnitts. Then

$$\begin{aligned} (r +_{\mathbb{R}} s) +_{\mathbb{R}} t &= \left\{ p +_{\mathbb{Q}} q \mid p \in r \text{ and } q \in s \right\} +_{\mathbb{R}} t \\ &= \left\{ \left(p +_{\mathbb{Q}} q\right) +_{\mathbb{Q}} u \mid p \in r,\ q \in s, \text{ and } u \in t \right\} \\ &= \left\{ p +_{\mathbb{Q}} \left(q +_{\mathbb{Q}} u\right) \mid p \in r,\ q \in s, \text{ and } u \in t \right\} \quad \text{[By the associativity of } +_{\mathbb{Q}}.] \\ &= r +_{\mathbb{R}} \left\{ q +_{\mathbb{Q}} u \mid q \in s \text{ and } u \in t \right\} \\ &= r +_{\mathbb{R}} (s +_{\mathbb{R}} t)\,, \end{aligned}$$

so $+_{\mathbb{R}}$ is associative. ■

We also defined $0_{\mathbb{R}} = \left\{ p \in \mathbb{Q} \mid p < 0_{\mathbb{Q}} \right\}$.

2. Show that for every $r \in \mathbb{R}$, $r +_{\mathbb{R}} 0_{\mathbb{R}} = r$. *[5]*

SOLUTION. It suffices to show that $r +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq r$ and $r \subseteq r +_{\mathbb{R}} 0_{\mathbb{R}}$.

Suppose that $p \in r +_{\mathbb{R}} 0_{\mathbb{R}}$. Then, by the definition of $+_{\mathbb{R}}$, $p = q +_{\mathbb{Q}} s$ for some $q \in r$ and some $s \in 0_{\mathbb{R}}$, *i.e.* $s <_{\mathbb{Q}} 0_{\mathbb{Q}}$. Then $p = q +_{\mathbb{Q}} s <_{\mathbb{Q}} q$, so, by the downward closure of r, $p \in r$. Thus $r +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq r$.

Now suppose that $p \in r$. Since r is a schnitt, it has no largest element, so there is some $q \in r$ with $p <_{\mathbb{Q}} q$. Let $s = p - q$. Then $s <_{\mathbb{Q}} 0_{\mathbb{Q}}$, so $s \in 0_{\mathbb{R}}$. It follows that $p = q + \mathbb{Q}\,(p - q) = q +_{\mathbb{Q}} s \in r +_{\mathbb{R}} 0_{\mathbb{R}}$. Thus $r \subseteq r +_{\mathbb{R}} 0_{\mathbb{R}}$.

Since $r +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq r$ and $r \subseteq r +_{\mathbb{R}} 0_{\mathbb{R}}$, it follows that $r +_{\mathbb{R}} 0_{\mathbb{R}} = r$. ■