Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Solutions to Assignment #1
Imaginary Matrices
Due on Friday, 12 September.

Before starting on this assignment, please read through the handout *Polyas Problem Solving Principles* and keep it in mind when working through problems **1**–**3**.

Recall that the complex numbers are basically the real numbers with a square root for -1, usually denoted by i, thrown in and then closed up under the usual arithmetic operations of addition and multiplication. A little more formally, the set of complex numbers is $\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$, with $+$ and $\cdot$ defined by v$(a+bi)+(c+di) = (a+c)+(b+d)i$ and $(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$. Note that his definition of multiplication gives us $i^2 = (0 + 1i)^2 = -1 + 0i = -1$. We also have that $\mathbb{R} = \{ a + bi \in \mathbb{C} \mid b = 0 \}$ is a subset of $\mathbb{C}$.

Let $\mathbf{M}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$ be the set of 2×2 matrices with entries from the real numbers, and let $\mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the 2×2 zero and identity matrices, respectively, in $\mathbf{M}_2(\mathbb{R})$.

1. Find a matrix $\mathbf{T} \in \mathbf{M}_2(\mathbb{R})$ such that $\mathbf{T}^2 = -\mathbf{I}_2$. *[2]*

SOLUTION 1. *Mess around a bit.* The simplest matrices that do the job put 0s on the diagonal and a 1 and a -1 on the antidiagonal, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. These two matrices are pretty easy to find by hit or miss; you can check at your leisure that both satisfy $\mathbf{T}^2 = -\mathbf{I}_2$. □

SOLUTION 2. *Be general and algebraic.* We try to solve the equation $\mathbf{T}^2 = -\mathbf{I}_2$, where $\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as generally as we can. This boils down to finding all the real numbers a, b, c, and d such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & ac+cd \\ ab+bd & bc+d^2 \end{bmatrix} = \begin{bmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Looking at the entries, this means that we need to find all the solutions to the system of equations $a^2 + bc = -1$, $c(a + d) = 0$, $b(a + d) = 0$, and $d^2 + bc = -1$.

Observe first that $c(a+d) = 0$ when $c = 0$ or $a+d = 0$, *i.e.* $d = -a$, and $b(a+d) = 0$ when $b = 0$ or $a + d = 0$. Note that if $a + d \neq 0$, we must have $b = c = 0$. However, we then have to have that $a^2 = d^2 = -1$, which is impossible for real numbers a and d. Thus it must be the case that $a + d = 0$, *i.e.* that $d = -a$. This cuts us down to just finding all the solutions to the equation $a^2 + bc = -1$. (Why?) If you think about it, this means that b can be any real number other than 0, and we get a solution by setting $c = -\dfrac{1+a^2}{b}$, which works no matter what real number a happens to be.

Thus a matrix $\mathbf{T}$ satisfying $\mathbf{T}^2 = -\mathbf{I}_2$ has the form $\mathbf{T} = \begin{bmatrix} a & b \\ -\left(1+a^2\right)/b & -a \end{bmatrix}$, where a and b can be any real numbers, as long as $b \neq 0$. For example, in the two matrices in Solution 1 above we have $a = 0$, with $b = -1$ and $b = 1$, respectively. ■

2. Explain why there is a copy of the complex numbers in $\mathbf{M}_2(\mathbb{R})$, with this copy using the addition and multiplication of matrices as its addition and multiplication. *[3]*

SOLUTION. The copy is easy to identify via the correspondence $a + bi \Longleftrightarrow a\mathbf{I}_2 + b\mathbf{T}$. For example, this corresponds 0 with $\mathbf{0}_2$, 1 with $\mathbf{I}_2$, and i with $\mathbf{T}$. We only need to check that the correspondence is maintained when using the appropriate addition and multiplication on each side of the correspondence. (That is, using the addition and multiplication of complex numbers on one side, and the addition and multiplication of 2×2 matrices on the other.)

Suppose that a, b, c, and d are any real numbers. By the definition of the correspondence, we have $a + bi \Longleftrightarrow a\mathbf{I}_2 + b\mathbf{T}$ and $c + di \Longleftrightarrow c\mathbf{I}_2 + d\mathbf{T}$. Then

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i \text{ and}$$
$$(a\mathbf{I}_2 + b\mathbf{T}) + (c\mathbf{I}_2 + d\mathbf{T}) = a\mathbf{I}_2 + c\mathbf{I}_2 + b\mathbf{T} + d\mathbf{T} = (a + c)\mathbf{I}_2 + (b + d)\mathbf{T},$$

and it is evident, by the definition of our correspondence, that $(a + c) + (b + d)i \Longleftrightarrow (a + c)\mathbf{I}_2 + (b + d)\mathbf{T}$. Thus the correspondence is maintained when using addition in the respective domains. Note that we implicitly used a lot of the algebraic properties of the respective domains without further ado, including associativity, commutativity, and distributivity of multiplication over addition.

Also,

$$\begin{aligned}
(a + bi)(c + di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\
&= (ac - bd) + (ad + bc)i \text{ and} \\
(a\mathbf{I}_2 + b\mathbf{T})(c\mathbf{I}_2 + d\mathbf{T}) &= ac\mathbf{I}_2^2 + ad\mathbf{I}_2\mathbf{T} + bc\mathbf{T}\mathbf{I}_2 + bd\mathbf{T}^2 \\
&= ac\mathbf{I}_2 + ad\mathbf{T} + bc\mathbf{T} - bd\mathbf{I}_2 \\
&= (ac - bd)\mathbf{I}_2 + (ad + bc)\mathbf{T},
\end{aligned}$$

so the correspondence is also maintained when using the multiplicationin the respective domains. Again, note that we implicitly used a lot of the algebraic properties of the respective domains.

It follows from all of the above that the set $\{\, a\mathbf{I}_2 + b\mathbf{T} \mid a,\, b \in \mathbb{R} \,\} \subseteq \mathbf{M}_2(\mathbb{R})$ is a copy of the complex numbers in $\mathbf{M}_2(\mathbb{R})$ via the correspondence $a + bi \Longleftrightarrow a\mathbf{I}_2 + b\mathbf{T}$. ■

The next step beyond the complex numbers are the *quaternions*, usually denoted by $\mathbb{H}$. They were invented/discovered in 1843 by William Rowan Hamilton (1805-1865), who used them to do things we mostly do with cross-products nowadays. To make the quaternions, you throw three different square roots of -1 – usually denoted by i, j, and k – into the real numbers which have a non-commutative multiplication among themselves. To be precise, we have:

$$\begin{array}{ccc}
i^2 = -1 & j^2 = -1 & k^2 = -1 \\
ij = k & jk = i & ki = j \\
ji = -k & kj = -i & ik = -j
\end{array}$$

Let $\mathbf{M}_4(\mathbb{R})$ be the set of 4×4 matrices with entries from thereal numbers, and let $\mathbf{O}_4$ and $\mathbf{I}_4$ be the 4×4 zero and identity matrices, respectively.

3. Find matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W} \in \mathbf{M}_4(\mathbb{R})$ such that:

$$\begin{array}{ccc} \mathbf{U}^2 = -\mathbf{I}_4 & \mathbf{V}^2 = -\mathbf{I}_4 & \mathbf{W}^2 = -\mathbf{I}_4 \\ \mathbf{UV} = \mathbf{W} & \mathbf{VW} = \mathbf{U} & \mathbf{WU} = \mathbf{V} \\ \mathbf{VU} = -\mathbf{W} & \mathbf{WV} = -\mathbf{U} & \mathbf{UW} = -\mathbf{V} \end{array} \quad [3]$$

Solution. The following matrices, among other possibilities, do the job:

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

These three matrices were found by hit or miss, with some inspiration drawn from the hit or miss approach taken in Solution 1 to question **1**. We leave it to the interested reader to check that these matrices satisfy the given equations. A fully general solution to this problem, along the lines of Solution 2 to question **1**, is a lot harder to achieve because there are so many variables. ($3 \times 16 = 48$ to start with ...) □

One could go on to use these matrices to show that there is a copy of the quaternions in $\mathbf{M}_4(\mathbb{R})$, but we'll save that as a possibility for another day. :-)

4. To what extent did your process in solving questions **1**–**3** follow the advice given in *Polyas Problem Solving Principles*? *[2]*

Solution. In answering each of questions **1**–**3**, it was necessary to understand the problem, which is Polya's first principle. (After all, if you don't understand the question, how do you know when you've answered it?)

As for Polya's second and third principles, to devise a plan and then execute it, in questions **1** and **3** I really had no plan except to tinker with matrices until I found some that worked, which I did. In question **3** the tinkedotsring was to some degree guided by the experience gained in answering question **1**, but that wasn't really a plan ... In question **2**, after working out the correspondence, the plan was simply to demonstrate that it was preserved under addition and multiplication in the two domains, which was a pretty straightforward set calculations to execute.

Looking back, Polya's fourth principle, in solving each question, was largely limited to fixing typos and making small improvements to phrasing.

Overall, the process I used conformed in part, but only in part, to Polya's advice. The only solution where the process conformed more-or-less fully to Polya's scheme was for question **2**. ■