More Reality

- infima, $+$, $\cdot$, & maybe other horrors

Recall:

- if $A \subset \mathbb{R}$ & A has an upperbound $u \in \mathbb{R}$, then the least upper bound or supremum of A is $\bigcup A = \bigcup_{a \in A} a$
- if $A \subset \mathbb{R}$ & has a lowerbound $l \in \mathbb{R}$, then the greatest lowerbound (or infimum) of A is the schnitt is $\bigcap A$ (if no largest element) where b is the largest element of $\bigcap A$ (if it has one)

$$(\bigcap A) \setminus \{B\} = \left(\bigcap_{a \in A} a\right) \setminus \{B\}$$

Definition: suppose $s, t \in \mathbb{R}$ (ie s & t are both schnitts), then $s+t$ is the schnitt:

$$s+t = \{p+q \mid p \in s, q \in t\}$$

$1_{\mathbb{R}} = \{p \in \mathbb{Q} \mid p < 1_{\mathbb{Q}}\}$

$2_{\mathbb{R}} = \{q \in \mathbb{Q} \mid q < 2_{\mathbb{Q}}\}$

- Check $s+t$ is a Schnitt is s, t are

(1) $s+t \neq \emptyset$ b/c there are $a \in s$, $b \in t$ $\Rightarrow$ $a+b \in s+t$

$s+t \neq \mathbb{Q}$ b/c there are $a \notin s$, $b \notin t$ $\Rightarrow$ $a+b \notin s+t$

(2) $s+t$ has no largest element:

suppose $p \in s+t$ so $p = a+b$ for some $a \in s$, $b \in t$

since s has no element, there is a $c \in s$ with $a<c$, & similarly there is a $d \in t$ such that $b<d$

then $p = a+b < c+d = q \in s+t$

(3) $s+t$ is downward closed:

suppose $p \in s+t$ & $q<p$

(why $q<p$?)

since $p \in s+t$, there are $a \in s$, $b \in t$ such that $p = a+b$ (we need $c \in s$, $d \in t$ such that $q = c+d$)

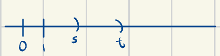
let $x = \frac{p-q}{2}$ $(2x = p-q)$ & then let:

$c = a-x$ & $d = b-x$ then

$$\begin{aligned} c+d &= (a-x)+(b-x) \\ &= (a+b) - 2x \\ &= p - 2\left(\tfrac{p-q}{2}\right) \\ &= q \end{aligned}$$

$\therefore$ $q \in s+t$ so $s+t$ is downward closed

How do we define multiplication on $\mathbb{R}$? (using schnitts)



$s \cdot t \overset{?}{=} \{ab \mid a \in s, b \in t\}$ is not a schnitt even if s, t are b/c by choosing $a<0$ in s & $b<0$ in t:

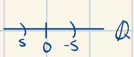
we can get any rational $q \geq 0$ as $q = a \cdot b$

- if $s \cdot t$ was a schnitt it wouldn't be a schnitt after all

→ fix? If $0<s$ & $0<t$ then define $s \cdot t = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{ab \mid a \in s$ & $b \in t$, $a<0$ & $b<0\}$

Q: how do we extend this to possibly non-positive schnitts?

1. given a schnitt $s \in \mathbb{R}$, how do we define "$-s$" as a schnitt?



$= \{p \mid p <_{\mathbb{Q}} 0\}$

$-s = 0_{\mathbb{R}} \setminus \{-p \mid p \in s, 0<p\}$ (if $q = \sup(s)$ is a rational)

2. we let $-s = 0_{\mathbb{R}} \cup \{0_{\mathbb{Q}}\} \cup \{-q \mid q <_{\mathbb{Q}} 0$ & $q \notin s\} \setminus \{-r\}$ if $s = \{q \in \mathbb{Q} \mid q<r\}$ for $r \in \mathbb{Q}$

Now we extend the definition of multiplication

s	t	$s \cdot t$
$+$	$+$	as before
$+$	$-$	$-(s \cdot (-t))$
$-$	$+$	$-((-s) \cdot t)$
$-$	$-$	$(-s) \cdot (-t)$

where we multiply positives