The Reals

or, as Stefan said "today we get real"

A Schnitt or Dedekind cut is a subset $r \subseteq \mathbb{Q}$ such that:

(1) $r \neq \emptyset$ & $r \neq \mathbb{Q}$

(2) r is "downward closed"

→ ie if $p \in r$ & $q < p$ $(p, q \in \mathbb{Q})$ then $q \in r$

(3) r has no largest element

→ ie if $p \in r$, there is a $q \in r$ such that $p < q$

→ eg: $0_{\mathbb{R}} = \{p \in \mathbb{Q} \mid p <_{\mathbb{Q}} 0\}$ $\quad \sqrt{2} = \{p \in \mathbb{Q} \mid p \leq_{\mathbb{Q}} 0\} \cup \{q \in \mathbb{Q} \mid q^2 <_{\mathbb{Q}} 2\}$

Definition

is a subset but not equal to

1. $\mathbb{R} = \{r \subsetneq \mathbb{Q} \mid r \text{ is a schnitt}\}$

2. $<_{\mathbb{R}}$ is given by $r <_{\mathbb{R}} s \iff r \subsetneq s$

→ Check $<_{\mathbb{R}}$ is a linear order

1) $<_{\mathbb{R}}$ is irreflexive

suppose r is a schnitt

$r = r \implies r \not\subsetneq r$

$\implies r \not<_{\mathbb{R}} r$

2) $<_{\mathbb{R}}$ is transitive

suppose r, s, t are schnitts & $r <_{\mathbb{R}} s$ & $s <_{\mathbb{R}} t$

[to show $r <_{\mathbb{R}} t$]

$r <_{\mathbb{R}} s <_{\mathbb{R}} t \implies r \subsetneq s \subsetneq t$

$\implies r \subsetneq t$

$\implies r <_{\mathbb{R}} t$

3) $<_{\mathbb{R}}$ satisfies trichotomy

Given schnitts s & r, we need to show that exactly one of $r <_{\mathbb{R}} s$, $r = s$, $s <_{\mathbb{R}} r$ is true.

→ if $r = s$, we're done ($r <_{\mathbb{R}} s$ & $s <_{\mathbb{R}} r$ are then not true)

→ if $r \neq s$, we need to show that $r <_{\mathbb{R}} s$ or $s <_{\mathbb{R}} r$:

since $r \neq s$, then either there is a $p \in r \setminus s$ or $q \in s \setminus r$ (or both

→ if $p \in r \setminus s$, then $q <_{\mathbb{Q}} p$ for all $q \in s$ b/c if there were some $q \in s$ such that $p \leq_{\mathbb{Q}} q$ then p would be in s by downward closure

→ ie $s \subsetneq r$

→ if $q \in s \setminus r$, then $r \subsetneq s$ in a similar way.

3. u is an upperbound for a set S if $s \leq u$ for all $s \in S$

→ w is a least upper bound (or a supremum) for S if:

(1) w is an upper bound for S &

(2) for every upperbound u for S, we have $w \leq u$

Theorem: if $S \subsetneq \mathbb{R}$ has an upperbound, it has a least upperbound

proof: suppose S is a subset of schnitts & u is a schnitt such that $s <_{\mathbb{R}} u$ for all $s \in S$

[u is an upperbound for S]

We claim that $w = \bigcup S = \bigcup_{s \in S} s$ is the least upperbound of S

→ first we need to show that w is a schnitt

(1) $\emptyset \neq s \subseteq w$ (for any $s \in S$), $w \neq \mathbb{Q}$ b/c $u \neq \mathbb{Q}$ so there is a $q \in \mathbb{Q} \setminus u$

then $p < q$ for all $p \in u$ & hence for all $p \in s \subseteq u$ for all $s \in S$

$\therefore$ $p \notin s$ for all $s \in S$ so $p \notin w = \bigcup_{s \in S} s$

(2) w is downward closed:

suppose $p \in w$ & $q <_{\mathbb{Q}} p$

→ if $p \in w = \bigcup_{s \in S} s$ then $p \in s$ for some $s \in S$

→ since it is downward closed, $q \in s \subseteq w$

$\therefore$ it is downward closed

(3) w has a greatest element:

suppose $p \in w$ [we need a $q \in w$ such that $p <_{\mathbb{Q}} q$]

then $p \in s$ for some $s \in S$ since s is a schnitt, it has no largest element & so there is some $q \in s$ such that $p <_{\mathbb{Q}} q$.

→ But then $q \in s \subseteq w$, so we're done.

→ thus w is a Schnitt.

→ it is an upperbound for S b/c if $s \in S$, then $s \subseteq \bigcup_{s \in S} s = w$ so $s \leq_{\mathbb{R}} w$ by definition

→ Why is w the least upper bound of S?

Suppose v is any upperbound for S (ie $s \leq_{\mathbb{R}} v$ for all $s \in S$)

⇒ since $s \leq v$ for all $s \in S$, $w = \bigcup_{s \in S} s \leq v$ so $w \leq_{\mathbb{R}} v$

} thus w is the least upperbound of S.