Embedding Countable Linear Orders into the rationals

} then the Schröder-Berstein Theorem

Theorem: Suppose A is a countable set & Δ is a linear order on A. Then there is a 1-1 function $(f: A \to \mathbb{Q})$ such that for all $a,b \in A$, $a \Delta b \Rightarrow f(a) < f(b)$

Proof: Since A is countable, we can enumerate it as $a_0, a_1, a_2, \dots$ [this list probably has nothing to do with Δ)

$\Rightarrow$ We build $f: A \to \mathbb{Q}$ recursively using this enumeration

let $f(a_0) = \frac{1}{2} = q_0$

given that $f(a_i) = q_i$ has be defined for $i = 0, \dots, k$ (w/ $a_i \Delta a_j \Rightarrow f(a_i) < f(a_j)$), we define $f(a_{k+1}) = q_{k+1}$ as follows:

- case (1): a_{k+1} is to the left of all $a_0, a_1, a_2, \dots, a_k$

  Then let $f(a_{k+1}) = q_{k+1}$ for any q_{k+1} that is to the left of all $q_0, q_1, \dots, q_k$

  ($\to$ we can also do this b/c the rationals don't have endpoints)

- case (2): a_{k+1} is to the right of all of $a_0, a_1, \dots, a_k$

  Then let $f(a_{k+1}) = q_{k+1}$ for any q_{k+1} that is to the right of all the $q_0, q_1, \dots, q_k$

- case (3): There are a_i & a_j which are to either side of a_{k+1} (& no others are closer), say: $a_i \Delta a_{k+1} \Delta a_j$

  Then let $f(a_{k+1}) = q_{k+1}$ for any q_{k+1} such that $q_i < q_{k+1} < q_j$

  (we can do this b/c there are infinitely many rationals btween any 2)

By the construction of $f: A \to \mathbb{Q}$, f is 1-1 & it preserves order

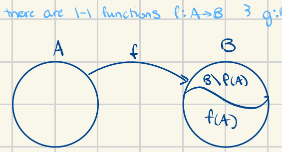
$\Rightarrow$ i.e. $a_i \Delta a_j \Rightarrow q_i = f(a_i) < f(a_j) = q_j$

The Schröder-Berstein Theorem

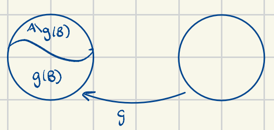
$\|A\| \le \|B\|$ & $\|B\| \le \|A\|$

$\Rightarrow \|A\| = \|B\|$

ie: if there are 1-1 functions $f: A \to B$ & $g: B \to A$, then there is a 1-1 onto function $h: A \to B$



$f(x) = \{f(x) \mid x \in X\}$



$\Rightarrow$ we'll divide A into Z & $A \setminus Z$

B into $f(Z)$ & $B \setminus f(Z)$ $\uparrow g \downarrow g^{-1}$

Then $h: A \to B$ will be defined by

$$h(a) = \begin{cases} f(a) & a \in Z \\ g^{-1}(a) & a \in A \setminus Z \end{cases}$$

[how do we pick $Z \subseteq A$ & prove it works)

Define $H: \mathcal{P}(A) \to \mathcal{P}(B)$ by $H(X) = A \setminus g(B \setminus f(X))$ $(X \subseteq A)$

then $Z = \{a \in A \mid \exists X \subseteq A \ni a \in X \subseteq H(X)\}$

* we'll show that:

1) $X \subseteq Y \subseteq A \Rightarrow H(X) \subseteq H(Y)$

2) $Z \subseteq H(Z)$

3) $g(B \setminus f(Z)) \subseteq A \setminus Z$

4) $g(B \setminus f(Z)) = A \setminus Z$

5) $h: A \to B$ is 1-1 & onto

1) $\mathcal{X} \subseteq \mathcal{Y} \subseteq A$

$\Rightarrow f(\mathcal{X}) \subseteq f(\mathcal{Y})$

$\Rightarrow B \setminus f(\mathcal{X}) \supseteq B \setminus f(\mathcal{Y})$

$\Rightarrow g(B \setminus f(\mathcal{X})) \supseteq g(B \setminus f(\mathcal{Y}))$

$\Rightarrow H(\mathcal{X}) = A \setminus g(B \setminus f(\mathcal{X})) \subseteq A \setminus g(B \setminus f(\mathcal{Y})) = H(\mathcal{Y})$

2) $Z \subseteq H(Z)$

$a \in Z \xRightarrow{\text{by definition}}$ for some $\mathcal{X} \subseteq A$, $a \in \mathcal{X} \subseteq H(\mathcal{X})$

* note that each $x \in \mathcal{X}$ also satisfies this (ie $x \in \mathcal{X} \subseteq H(\mathcal{X})$) (ie $x \subseteq Z$)

$\therefore a \in \mathcal{X} \subseteq H(\mathcal{X}) \subseteq H(Z)$ so $a \in H(Z)$

$\therefore Z \subseteq H(Z)$

3) $g(B \setminus f(Z)) \subseteq A \setminus Z$

$\Rightarrow$ Suppose, by way of contradiction, that $b \in B \setminus f(Z)$ but $g(b) = a \in Z$

then, by definition, $a \in \mathcal{X} \subseteq H(\mathcal{X}) \subseteq H(Z)$

$\therefore a \in H(Z) = A \setminus g(B \setminus f(Z))$

$\therefore b \notin B / f(Z)$

4) $g: B \setminus f(Z) \to A \setminus Z$ is onto (so $g(B \setminus f(Z)) = A(Z)$)

Suppose $a \in A \setminus Z$ [to show: $a = g(b)$ for some $b \in B \setminus f(Z)$]

Since $a \notin Z$, $Z \cup \{a\} \not\subseteq H(Z \cup \{a\})$ (since otherwise $a \in Z$ by definition)

$Z \subseteq Z \cup \{a\} \Rightarrow Z \subseteq H(Z) \subseteq H(Z \cup \{a\})$

$\Rightarrow a \notin H(Z \cup \{a\})$

$\Rightarrow a \notin H(Z)$

ie: $a \notin H(Z) = A \setminus g(B \setminus f(Z))$

$\Rightarrow a \in g(B \setminus f(Z))$

$\Rightarrow a = g(b)$ for some $b \in B \setminus f(Z)$ as desired

Define $h: A \to B$ by

$$h(a) = \begin{cases} f(a) & a \in Z \\ g^{-1}(a) & a \in A \end{cases}$$

is 1-1 onto $A \to B$ b/c f is 1-1 onto on $Z \to f(Z)$ & g is 1-1 onto on $B \setminus f(Z) \to A \setminus Z$ so g^{-1} is 1-1 onto on $A \setminus Z \to B \setminus f(Z)$