Cardinality

or: the sizes of sets (& finite vs infinite along the way)

A , B

Q: when are these the same size?

Answer: when the same number of elements

→ ie: there is a 1-1 onto function (a "bijection")

$f: A \to B$

→ eg: $\mathbb{N}$ 0   1   2   3   4

$\mathbb{Z}$ ... -3   -2   -1   0   1   2   3 ...

Claim: they are the same size, or the same cardinality

→ Pair of $\mathbb{N}$ & $\mathbb{Z}$ as follows

1. if $n \in \mathbb{N}$ is even $(n=2k)$, pair it off with $k \in \mathbb{Z}$
2. if $n \in \mathbb{N}$ is odd $(n=2k+1)$, pair it off with $-(k+1) \in \mathbb{Z}$

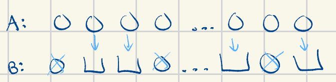
→ This is 1-1 & onto

A set is infinite if it is the same size as some proper subset of itself

→ otherwise, it's finite

a "proper" subset (a subset but not equal to)

suppose A is a finite set & $B \overset{\subset}{\neq} A$. why can't there be a 1-1 onto map $A \to B$



don't all fit

→ Pigeonhole principle: if you distribute some finite number of objects into a smaller collection of buckets, some bucket will get more than one object

Claim: $\|\mathbb{N}\| = \|\mathbb{Q}\|$

cardinality of $\mathbb{N}$ cardinality of $\mathbb{Q}$

Proof: we need to show there is a 1-1 onto function: $f: \mathbb{N} = \mathbb{Q}$

$f(n) = \begin{cases} 1/n & n \neq 0 \\ 0 & n = 0 \end{cases}$ is 1-1 but not onto ($\frac{2}{3}$ is not converted

→ why it doesn't work:

$$\begin{matrix} \cdots & \frac{-3}{1} & \frac{-2}{1} & \frac{-1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \cdots \\ \cdots & \frac{-3}{2} & \frac{-2}{2} & \frac{-1}{2} & \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \cdots \\ \cdots & \frac{-3}{3} & \frac{-2}{3} & \frac{-1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \cdots \end{matrix}$$

* traversing this matrix & skipping things equal to what you already seen gives a 1-1 onto map b/ween $\mathbb{N}$ & $\mathbb{Q}$

A set equal to $\mathbb{N}$ in size is said to be <u>countable</u> or <u>countably infinite</u>

$\|A\| < \|B\|$ means there is a 1-1 function $f: A \to B$, but no "onto" function

$\|A\| \leq \|B\|$ means $\|A\| < \|B\|$ or $\|A\| = \|B\|$

→ Schröder - Bernstein Theorem

$\|A\| \leq \|B\|$ & $\|B\| \leq \|A\| \Rightarrow \|A\| = \|B\|$

Theorem: Suppose $\lhd$ is a linear order of a countable set A. Then there is a 1-1 function $\delta: A \to \mathbb{Q}$ that preserves order

→ ie: $a \lhd b \Rightarrow \delta(a) <_{\mathbb{Q}} \delta(b)$

Proof: By induction (on?)