The Rationals

Intuition: $\frac{a}{b} \overset{?}{=} \frac{c}{d}$ $\iff$ $ad = bc$

Consider $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{(a,b) \mid a,b \in \mathbb{Z}$ & $b \neq 0\}$

- we'll define a relation $\approx$ by $(a,b) \approx (c,d) \iff ad = bc$

Claim: $\approx$ is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$

Pf: (1) $\approx$ is reflexive b/c $ab = ba$

so $(a,b) \approx (a,b)$

(2) $\approx$ is symmetric b/c if $ad = bc$ then $cb = da$

so $(a,b) \approx (c,d) \Rightarrow (c,d) \approx (a,b)$

(3) $\approx$ is transitive

- suppose $(a,b) \approx (c,d)$ & $(c,d) \approx (e,f)$ [need to show $(a,b) \approx (e,f)$]

then $ad = bc$ & $cf = de$

so $adcf = bcde$

thus $af = be$ by

1. case 1: $a = c = 0$ & $c = e = 0$ but then $af = be = 0$
2. case 2: $a, c, e \neq 0$

- Have $af(cd) = be(cd)$ since also $d \neq 0$, $cd \neq 0$
- so by the cancellation law for multiplication on $\mathbb{Z}$, $af = be$ ... //

intuitively, this represents the ratio $\frac{a}{b}$

- Then $\mathbb{Q} = \{[(a,b)]_{\approx} \mid a,b \in \mathbb{Z}$ & $b \neq 0\}$

Definition:

1. $\cdot_{\mathbb{Q}}$ is defined by $[(a,b)]_{\approx} \cdot_{\mathbb{Q}} [(c,d)]_{\approx} = [(ac, bd)]_{\approx}$
$\left(\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}\right)$

2. $+_{\mathbb{Q}}$ is defined by $[(a,b)]_{\approx} +_{\mathbb{Q}} [(c,d)]_{\approx} = [(ad + cb, bd)]_{\approx}$
$\left(\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}\right)$

We ought to show $+_{\mathbb{Q}}$ & $\cdot_{\mathbb{Q}}$ are "well defined" (i.e Stefan's too lazy & it's boring)

$$\frac{[(a,b)]_{\approx}}{[(c,d)]_{\approx}} = [(ad, bc)]_{\approx} = \frac{ad}{bc}$$

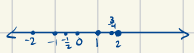
$\left(\frac{a/b}{c/d}\right)$ $\neq 0_{\mathbb{Z}}$ so $c \neq 0$

$<_{\mathbb{Q}}$? $[(a,b)]_{\approx} <_{\mathbb{Q}} [(c,d)]_{\approx}$

$\iff ad <_{\mathbb{Z}} cd$ which gives a linear order

$\mathbb{N}$ 0 1 2 ...

$\mathbb{Z}$ ... −2 −1 0 1 2 ...

$\mathbb{Q}$ 

$<_{\mathbb{Q}}$ has no endpoints: a rational between any 2 others (actually infinitely many)