Multiplication

∗ w/ a bonus of the Right Cancellation Law (for addition)

Definition of multiplication

→ $n \cdot 0 = 0$ for all n

→ given that $n \cdot k$ has been defined, $n \cdot S(k) = n \cdot (k+1)$

$= nk + n$

Claim: "$\cdot$" is associative (ie: $(n \cdot m) \cdot k = n \cdot (m \cdot k)$)

→ Proof by induction on k

Base Step $k=0$ for all $n, m \in \mathbb{N}$

$(n \cdot m) \cdot 0 = 0$ by definition

$n \cdot (m \cdot 0) =$

$n \cdot 0 =$

$0 = 0$

Induction Hypothesis $k = \ell$ for all $m, n \in \mathbb{N}$

$(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$

Inductive Step $k = \ell \Rightarrow k = \ell + 1 = S(\ell)$

want to show: $(n \cdot m) \cdot S(\ell) = n \cdot (m \cdot S(\ell))$

$(n \cdot m) \cdot S(\ell) = n \cdot (m \cdot S(\ell))$ by definition

$= n \cdot (m \cdot \ell + m)$ by definition

$= (n \cdot m) \cdot \ell + n \cdot m$

$= n \cdot m + (n \cdot m) \cdot \ell$

$= n \cdot m + n \cdot (m \cdot \ell)$ by I.H.

$= n \cdot m \cdot (1) + n \cdot (m \cdot \ell)$

$= n \cdot (m \cdot 1) + n \cdot (m \cdot \ell)$

Subclaim: $(n \cdot m) \cdot 1 = n \cdot (m \cdot 1)$

b/c $n \cdot 1 = n \cdot S(0)$

$= n \cdot 0 + n$

$= 0 + n$

$= n$

Show $(n \cdot m) \cdot 2 = n \cdot (m \cdot 2)$

$(n \cdot m) \cdot 2 = n \cdot m \cdot 1 + n \cdot m$

$n \cdot (m \cdot 2) = n \cdot (m \cdot S(1))$

$= n(m \cdot 1 + m)$

We have: $(n \cdot m) \cdot 0 = n \cdot (m \cdot 0)$

$(n \cdot m) \cdot 1 = n \cdot (m \cdot 1) \Rightarrow (n \cdot m) \cdot S(0) = n \cdot (m \cdot S(0))$

We want: $(n \cdot m) \cdot 2 = n \cdot (m \cdot 2)$

$\boxed{\Rightarrow (n \cdot m) \cdot S(1) = n \cdot (m \cdot S(1))}$

$\Rightarrow (n \cdot m) \cdot S(S(0)) = n \cdot (m \cdot S(S(0)))$

$(n \cdot m) \cdot S(1) = (n \cdot m) \cdot 1 + n \cdot m$

$n \cdot (m \cdot S(1)) = n \cdot (m + 1 + m)$

to be continued Tuesday