The Axioms for Sets

} what we can do with them

Already Covered for the Zermelo-Fraenkel Axioms so far:

1. Empty Set Axiom
   - → "There is an empty set"
   - → $\exists x \forall y \ (\neg y \in x)$
2. Union Axiom
   - → "If x is a set, then $\cup x = \bigcup_{a \in x} a = \{y \mid \exists a (y \in a \wedge a \in x)\}$ is a set" ($\wedge$ = and)
3. Comprehension Axiom
   - → "If x is a set & $\phi(y)$ is a formula with y free, then $\{y \in x \mid \phi(y)\}$ is a set"

Today

The Zermelo-Fraenkel Axioms (continued)

→ Let $R = \{x \in U \mid \neg(x \in x)\}$

- → Pair Set Axiom: "If x & y are sets, then $\{x, y\}$ is a set too" ✱ a surprisingly powerful axiom
- → Power Set Axiom: "If x is a set, the $\mathcal{P}(x) = \{x \mid y \subseteq x\}$ is a set too"
- → Axiom of Foundation (or Regularity): "If x not empty, there is some element $y \in x$ such that $x \cap y \neq \emptyset$"
- → Axiom of Replacement: suppose x is a set & $\phi(y,z)$ is a formula which defines a function with its domain in x.

  Then, $\{z \mid y \in x \wedge \phi(y,z)\}$ ($\wedge$ = and)
- → Axiom of Extensionality: "If x & y are sets, then x & y are equal exactly when the have the same elements"
  - → $\forall x \forall y \, (x = y \iff \forall z \, (z \in x \iff z \in y))$
- → Axiom of Infinity:
  - → Let S be the following operation on sets:

    $S(x) = x \cup \{x\}$

    Why is $S(x)$ defined for all sets x given the other axioms?

    → If x is a set... then $\{x, x\}$ is a set (Pair)

    $\{x, x\} = \{x\}$ by extensionality & is also a set

    → then $\{x, \{x\}\}$ is a set (Pair)

    by Union, $\bigcup \{x, \{x\}\} = x \cup \{x\}$ is a set too

  - → Then $\omega = \{\emptyset, S(\emptyset), S(S(\emptyset)), \ldots\}$ is a set (empty set; $\mathbb{N}$ =)

    it gets complicated quickly — Aligns w/ the natural numbers

    $S(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\}$ — $\emptyset = 0$, $= 1$

    $S(S(\emptyset)) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$ — $= 2$

    $S(S(S(\emptyset))) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ — $= 3$
- → Axiom of Choice: "If x is a set of non-empty sets, then there is a function $f : x \to \cup x$ such that for all $y \in x$, $f(y) \in y$"
  - → rarely need the full power of this

The Natural Numbers ($\mathbb{N}$)

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$0 = \emptyset$ ✱ think of them as a collection of the predecessors

$1 = S(0) = \{0\}$ (successor of 0)

$2 = S(1) = \{0, 1\}$ (successor of 1)

$3 = S(2) = \{0, 1, 2\}$

etc

→ how do we define addition on $\mathbb{N}$?

→ $n+0=n$ for all $n \in \mathbb{N}$

→ given that $n+k$ has been defined, $n+S(k) = S(n+k)$ — think of this as $n+(k+1)$

→ verify $2+2=4$:

$2+2 = 2+S(1)$

$= S(2+1)$

$= S(2+S(0))$

$= S(S(2+0))$

$= S(S(2))$

$= S(3)$

$= 4$

Claim 1: $+$ is associative

for all $k,n,m \in \mathbb{N}$, $(k+n)+m = k+(n+m)$

→ proof by induction on m

Base Step: $(m=0)$ for all $k,n \in \mathbb{N}$,

$(k+n)+0 = k+n$ by definition of addition

$= k+(n+0)$ by definition of addition

Induction Hypothesis: $(m=\ell)$ for all $n,k \in \mathbb{N}$,

$(k+n)+\ell = k+(n+\ell)$

Inductive Step: $(m=\ell \rightarrow m=(\ell+1))$ assume the IH

$(k+n)+(\ell+1) = (k+n)+S(\ell)$

$= S((k+n)+\ell)$ by definition

$= S(k+(n+\ell))$ by I.H.

$= k+S(n+\ell)$ by definition

$= k+(n+S(\ell))$ by definition

$= k+(n+(\ell+1))$ by definition

∴ by induction, addition is associative.

Claim 2: $+$ is commutative

for all $k,n,m \in \mathbb{N}$, $n+k = k+n$

→ proof by induction on n

Base Step: $(n=0)$ for all k,

$0+k = k+0$

→ By the Lemma, $0+k = k+0$

Induction Hypothesis: $(n=\ell)$ for all k,

$\ell+k = k+\ell$

Inductive Step: $(n=\ell \rightarrow n=\ell+1)$

$(\ell+1)+k = S(\ell)+k$

we need to verify that $S(\ell)+k = S(\ell+k)$

$= S(\ell+k)$ by Lemma 2

$= S(k+\ell)$ by I.H.

$= k+S(\ell)$ by definition.

$= k+(\ell+1)$

Lemma 1: $0+k = k+0$ for all $k \in \mathbb{N}$

proof by induction of k

Base Step: $(k=0)$ $0+0=0$ by definition

Induction Hypothesis: $(k=m)$ $0+m=m$

Inductive Step: $(k=m \rightarrow k=m+1)$

$0+(m+1) = 0+S(m)$

$= S(0+m)$ by definition

$= S(m)$

$= (m+1)$

$= (m+1)+0$

Lemma 2: $S(\ell)+k = S(\ell+k)$ for all ℓ & k

proof by induction on k

Base Step: $(k=0)$

$S(\ell)+0 = S(\ell)$ by definition of addition

$= S(\ell+0)$ by definition of addition

Induction Hypothesis: $(k=m)$

$S(\ell)+m = S(\ell+m)$

Inductive Step: $(k=m \rightarrow k=m+1)$

$S(\ell)+(m+1) = S(\ell)+S(m)$

$= S(S(\ell)+m)$ by definition

$= S(S(\ell+m))$ by I.H.

$= S(\ell+S(m))$

$= S(\ell+(m+1))$