Axioms for Sets

Recap

Symbols:

	Formally	Unofficially
quantifiers	$\forall$ ("for all")	$\exists$ ("there is") → actually an abbreviation for $\neg\forall\neg$
connectives	$\neg$, $\rightarrow$	$x_0, x_1, x_2, \ldots$
variables		

Grouping symbols: (,)

Equality: $=$ (the necessary relation)

Binary relation: $\in$ ("is an element of")

→ informally: $\cup$ (union)

$\cap$ (intersection)

$\setminus$ (relative component)

$\subseteq$ (subset)

Formulas

1. $x_k = x_m$ & $x_i \in x_j$ are the atomic formulas
2. α is a formula $\Rightarrow$ $(\neg\alpha)$ is also one
3. α & β are formulas $\Rightarrow$ $(\alpha \rightarrow \beta)$ is also one
4. α is a formula $\Rightarrow$ $\forall x_i\, \alpha$ is also one
5. nothing else is a formula

Today

Rule of Procedure: If you have $(\alpha \rightarrow \beta)$ and α, you can infer β

Logical Axiom Schema

A1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$

A2. $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$

A3. $(((\neg\beta) \rightarrow (\neg\alpha)) \rightarrow (((\neg\beta) \rightarrow \alpha) \rightarrow \beta))$

A4. $(\forall x \alpha \rightarrow \alpha_y^x)$

→ where you can replace every free x in α by y, as long as the y doesn't get captured by a quantifier in α

A5. $(\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta))$

A6. $(\alpha \rightarrow \forall x \alpha)$

→ so long as x does not occur free in α

A7. $x = x$

→ for any variable x

A8. $x = y \rightarrow (\alpha \rightarrow \beta)$

→ where α & β are atomic & you replace some - or all - instances of x in α by y to get β

We will show that for any formula α & any variable x, that $\{\alpha\} \vdash \exists x \alpha$

1. $((\forall x(\neg\alpha) \rightarrow (\neg\alpha)) \rightarrow (\alpha \rightarrow \overbrace{\forall x(\neg\alpha)}^{\text{equivalent to } \exists x\alpha}))$ by (In propositional logic, we can show that) $\vdash ((\beta \rightarrow (\neg\gamma)) \rightarrow (\gamma \rightarrow (\neg\beta)))$
2. $(\forall x(\neg\alpha) \rightarrow (\neg\alpha))$ by A4 since $(\neg\alpha)_x^x$ is $\neg\alpha$ (have anti-cedent so can apply MP)
3. $(\alpha \rightarrow (\neg\forall x(\neg\alpha)))$ by 1,2 MP
4. α by hypothesis/premis (can site the hypothesis whenever needed)
5. $(\neg\forall x(\neg x))$ by 3,4 MP (have anti-cedent so can apply MP)
6. $\exists x \alpha$ by the definition of $\exists$

Zermelo-Fraenkel Set Theory with the Axiom of Choice

→ also called ZFC

- Suppose there is a universal set
  - → ie: a set of which every set is an element
  - → we will call this U

Let $R = \{x \in U \mid \neg(x \in x)\}$ ※ the Barber Shop paradox

→ Is $R \in R$? would need to satisfy $\neg(R \in R)$

→ <u>Union Axiom</u>: if x is a (non-empty) set of sets, then $\bigcup x = \bigcup_a = \{y \mid y \in a \text{ for some } a \in x\}$ is a set

  - → $\emptyset = \bigcup \emptyset$

→ <u>Empty Set Axiom</u>: the empty set, $\emptyset$, is a set

→ <u>Comprehension Axiom</u>: If x is a set & φ is a formula with one free variable, y, then $\{y \in x \mid \varphi(y)\}$ is also a set