The Soundness Theorem

$(\Sigma \vdash \varphi \Rightarrow \Sigma \vDash \varphi)$

ie: if φ is deducible from Σ, then whenever every statement in Σ is true, so is φ

Proof

Suppose $\Sigma \vdash \varphi$, ie there is a deduction $\psi_1, \psi_2, \ldots, \psi_n$ where each ψ_i is:

(1) a logical axiom

(2) a hypothesis (ie $\psi \in \Sigma$)

or

(3) it follows from some ψ_j, ψ_k (with $j,k < i$) by MP

We'll use induction on n (the length of the deduction) to show that if every statement in Σ is true, so is ψ_n

Base Step: $(n=1)$

→ In this case ψ_1 is the deduction

→ suppose that every statement in Σ is true (for some assignment of T/F to each atomic formula occuring in a formula in Σ)

→ Then ψ_1 is true b/c either $\psi_i \in \Sigma$ (so it's true) or it's a logical axiom (so it's true)

Inductive Hypothesis: $(n=k)$

→ If $\psi_1, \ldots, \psi_k$ is a deduction from Σ

→ then whenever every statement in Σ is true, so is ψ

Inductive Step: $(n=k+1)$

→ so our deduction is $\psi_1, \ldots, \psi_k, \psi_{k+1}$

(to show: if every statement in Σ is true, then so is ψ_{k+1})

→ then ψ_{k+1} is:

(1) a logical axiom

(2) $\in \Sigma$

or

(3) follows from ψ_i, ψ_j $(i,j < k+1)$ by MP

→ then if every statement in Σ is true (cases (1) & (2)), then ψ_{k+1} is true if it is an axiom or in Σ

→ in case (3), where ψ_k follows from ψ_i & ψ_j by MP, by the Induction Hypothesis, since $i,j \leq k < k+1$, ψ_i & ψ_j are true whenever every statement in Σ is true

→ since ψ_{k+1} follows from ψ_i & ψ_j by MP, we either have ψ_i is $\psi_j \to \psi_{k+1}$ OR ψ_j is $\psi_i \to \psi_{k+1}$

→ if ψ_i is $\psi_j \to \psi_{k+1}$ & both ψ_i & $\psi_j \to \psi_{k+1}$ are true, then ψ_{k+1} is true by how "→" works

→ Similarly for every other case

∴ If the Induction Hypothesis is true, the Inductive Step works

→ By mathematical induction, if $\Sigma \vdash \varphi$, then $\Sigma \vDash \varphi$ is correct //

Another example of Induction

"Every official formula of propositional logic has exactly as many left as right parentheses"

Proof:

We will proceed by induction on the number of connectives in the formula

Base Step: $(n=0)$

→ then the formula is atomic, so it has no parentheses & hence an equal number of left & right parentheses

Induction Hypothesis: $(n=k)$

→ every formula with $\leq k$ connectives, has an equal number of left & right parentheses

Inductive Step: $(n=k+1)$

→ suppose the formula α has $k+1$ connectives

→ then α is either $(\neg\beta)$ for some formula β or α is $(\gamma \to \delta)$ for some formulas γ & δ

(1) If α is $(\neg\beta)$, then β has $(k+1)-1=k$ connectives

→ By the I.H., β has the same # of left as right parentheses, we'll call this number b.

→ then α has $1+b$ left parentheses & $b+1$ right parentheses (hence the same number of each).

(2) If α is $(\gamma \to \delta)$, then γ & δ <u>together</u> have k connectives

→ By the IH since γ & δ each have to have $\leq k$ connectives, they each have the same number of left as right parentheses, say g & d respectively

→ then α (which is $(\gamma \to \delta)$) has $1+g+d$ left parentheses & $g+d+1$ right parentheses, hence an equal number of each.

→ thus, α has the same number of left & right parentheses

∴ by induction, every official formula has the same number of left & right parentheses //

Propositional logic <u>does not</u> handle quantifiers:

→ "for all" $\forall$

→ "there exists" $\exists$

A "first-order" language needs:

- quantifier(s): $\forall$, $\exists$
- variables: $x, y, z, \ldots$ later end of the alphabet
  → $(x_0, x_1, x_2, \ldots)$ more formally
- functions: f, g, h middle of the alphabet
  → f_n^k (k → # of places of input; n → subscript)
  → $(+, -, \times, \div)$
- connectives: $(\to, \neg, \wedge, \vee, \leftrightarrow)$
- grouping symbols: (,)
- constants: $a, b, c, \ldots$
  → $c_0, c_1, c_2, \ldots$ more formally
  → example: $0, 1, \sqrt{2}, e, 2 \ldots$
- relations: P, Q, R
  → example: $\in, <, >, \ldots$
  → R_n^k (k → # of places)

(won't do higher than first order languages)

A (minimal) language for set theory

- quantifier: $\forall$
  → $\exists x \varphi$ shorthand for $(\neg \forall x (\neg \varphi))$
- variables: $x_0, x_1, x_2, \ldots$
  → x, y, z
- grouping symbols: (,)
- relations: $\in$
  → $\cup, \cap, \setminus, \subseteq$
- equality: $=$
  → screwed if we can't tell when two things are equal
- functions: <u>none</u> $(\emptyset)$
- constants: <u>none</u> $(\emptyset)$
- connectives: $\neg, \to$
  → $\wedge, \vee, \leftrightarrow$

The formulas of the language are:

(1) the "atomic" formulas are: $x \in y$ & $y \in x$

(2) if α & β are formulas, so are $(\neg\alpha)$ & $(\alpha \to \beta)$

(3) if α is a formula, so is $\forall x \alpha$ → for any variables

(4) nothing else is a formula