Propositional Logic

(And some first-order logic afterwards)

Propositional Logic So Far:

- Symbols
  1. atomic formulas
     a. official: $A_1, A_1, A_2, \ldots$
     b. unofficial: $A, B, C, \ldots$
  2. 2 connectives
     a. official: $\neg, \rightarrow$
     b. unofficial: (also includes) $\wedge, \vee, \leftrightarrow$
  3. grouping symbols: (,)

- a string of symbols is a formula if:
  1. it is an atomic formula
  2. it is $(\neg\alpha)$, where α is a formula
  3. it is $(\alpha \rightarrow \beta)$, where α & β are formulas
  4. nothing else is a formula

- our only rule of procedure is <u>Modus Ponens</u> (MP)
  - given α & $(\alpha \rightarrow \beta)$, we may infer β

- logical axioms schema
  1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$
  2. $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$
  3. $(((\neg\beta) \rightarrow (\neg\alpha)) \rightarrow (((\neg\beta) \rightarrow \alpha) \rightarrow \beta))$

Deductions

definition: (from a set of hypotheses Σ) is a sequence of formulas: $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n$ such that every formula φ_k in the deduction is either:

1. a logical axiom

or

2. a hypothesis $(\varphi_k \in \Sigma)$

or

3. follows from preceding formulas φ_i & φ_j $(i, j < k)$ by MP
   - φ_i is $(\varphi_j \rightarrow \varphi_k)$ or φ_j is $(\varphi_i \rightarrow \varphi_k)$

We'll try to prove $(\varphi \rightarrow \varphi)$ from $\Sigma = \emptyset$ $(\Sigma \vdash (\varphi \rightarrow \varphi))$

-> options to try

1. $(\varphi \rightarrow (\varphi \rightarrow \varphi))$ Axiom Schema 1
2. $((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi)))$ Axiom Schema 2
3. $((\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi))$ 1, 2 MP
   - this is a deduction of $(\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi)$ from no hypotheses

try again:

2. $((\overset{\alpha}{\varphi} \rightarrow (\overset{\beta}{(\varphi \rightarrow \varphi)} \rightarrow \overset{\gamma}{\varphi})) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)))$ Axiom 2
3. $(\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi))$ Axiom 1
4. $((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))$ 2, 3 MP
5. $(\varphi \rightarrow \varphi)$ 1, 4 MP

<u>Deduction Theorem</u>

If Σ (→ the hypotheses) is a set of formulas & α & β are formulas, then:

$\Sigma \vdash (\alpha \rightarrow \beta)$ (There is a deduction of $(\alpha \rightarrow \beta)$ from the set of hypotheses Σ)

if, and only if, $\Sigma \cup \{\alpha\} \vdash \beta$

- redo the proof above:

$\emptyset \vdash (\varphi \rightarrow \varphi)$

<=> $\{\varphi\} \vdash \varphi$

1. φ hypothesis

<u>Soundness Theorem</u>

("deductions preserve truth")

If $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$

"Σ entails α" (every assignment of T & F to atomic formulas that make every formula in Σ true, also makes α true)

Proof

$\Sigma \vdash \alpha$ means that there is a deduction $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that each φ_i is:

1. in Σ
2. a logical axiom
3. follows from preceding formulas by MP

We'll prove this by induction

Type I $<$ Type II $\leq$

aside: If you have a statement involving $n \in \mathbb{N}$ & it is true for $n=0$ (or $n=1$) & if it is true for $n \leq k$ implies it is true for $n=k+1$ then it is true for all $n \geq 0$ (≥ 1)