Baby Set Theory

09 Sept 25

or - What can we do with $\{,\}, \in, \cap, \cup, \setminus$?

Challenge: Define (a,b) using only $\{,\}$.

$\{a,b\} = \{b,a\}$ * sets don't have order

→ why doesn't $\{a, \{b\}\}$ work? * what if $a = \{c\}$, then its unclear which comes first

→ one solution

There are two special objects ($*$ & O)

then: $(a,b)' = \{\{*,a\}, \{O,b\}\}$

→ Kuratowski (1920s)

$(a,b) = \{\{a\}, \{a,b\}\}$

$(a,a) = \{\{a\}, \{a,a\}\}$

$= \{\{a\}, \{a\}\}$

$= \{\{a\}\}$

Given that we have ordered pairs, we can have sets of them:

sets: A, B

→ $\{(a,b) \mid a \in A,\ b \in B\}$

→ $A \times B$ = "A cross B"

ie: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

Things we can define as collections of ordered pairs?

· examples: 1. $f: A \to B$ is a collection of ordered pairs from $A \times B$

→ $f \subseteq A \times B$ & $f(a) = b$ means $(a,b) \in f$

→ example: (identity map) $I: \mathbb{R} \to \mathbb{R}$ is $I = \{(a,a) \mid a \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$

What makes a collection $f \subseteq A \times B$ a function $f: A \to B$?

→ "For each $a \in A$, there is at most one $b \in B$ such that $f(a) = b$"

$[(a,b) \in F]$

What makes a function:

i) 1 to 1? (injective) → for each $a \in A$, there is an unique $b \in B$ such that $f(a) = b$ & if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

ii) "onto"? (surjective) → for $f: A \to B$ to be "onto" is if for all $b \in B$ there is some $a \in A$ such that $f(a) = b$

→ "b" can have more than one "a"

Notation

1. Image of f ($\mathrm{Im}(f)$)

$f''(A) = f(A) = \{b \in B \mid f(a) = b \text{ for some } a \in A\}$

2. Domain of f ($\mathrm{dom}(f)$)

$f^{-1}(B) = f^{-1}B = \{a \in A \mid f(a) \in B\}$ (ie f is defined on "a")

→ eg: $\ln: \mathbb{R} \to \mathbb{R}$ (natural log)

$\mathrm{Im}(\ln) = \mathbb{R}$

$\mathrm{dom}(\ln) = (0, \infty)$

What besides functions $A \to B$, can be thought of as subsets of $A \times B$?

→ we can define binary relations using ordered pairs

→ a relation is any subset $R \subseteq A \times B$

→ eg: "less than" ($\_ < \_$ on $\mathbb{R}$)

- $< \subseteq \mathbb{R} \times \mathbb{R}$ & $a < b$ means $(a,b) \in <$

→ How can we define the particular relation $<$ on $\mathbb{R}$?

$a < b$ iff: $a < b \iff 0 < b - a$

$\iff$ there is some $r \in \mathbb{R}$ such that $r^2 = b - a$ & $r \neq 0$

$\iff \exists\, r \in \mathbb{R} \setminus \{0\} : r^2 = b - a$

("$\exists$" is "there exists")

("$\setminus$" is a relative compliment. $\{a \in A \mid a \notin B\}$)

Definition:

A linear order on A is a binary relation $\lhd \subseteq A \times A$ satisfying certain conditions:

i) Antisymmetry: for all $a, b \in A$, if $a \lhd b$, then not $b \lhd a$ — * don't need once we have iii & iv. (redundant)

ii) Irreflexivity: for all $a \in A$, not $a \lhd a$ — * don't need once we have iii & iv. (redundant)

iii) Transitivity: for all $a, b, c \in A$, if $a \lhd b$ & $b \lhd c$ then $a \lhd c$

iv) Trichotomy: for all $a, b \in A$, exactly one of $a \lhd b$, $b \lhd a$, or $a = b$ is true

Propositional Logic

Logic of Connections — * does not cover quantifiers "for all" & "there exists"

$\neg$ = not

$\vee$ = or

$\wedge$ = and

$\rightarrow$ = if... then...

$\leftrightarrow$ = if and only if

The moon is made of smelly cheese.
Smelly cheese has a smell

The moon has a smell

← by propositional logic

↳ becomes

moon → smelly cheese
smelly cheese → smell
---
moon → smell

→ verify using a truth table

Moon	Smelly Cheese	has a smell	M → Smy	Smy → Sm	M → Sm
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	T
F	T	F	T	T	T
F	F	F	T	T	T