Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Final Examination
Due on Friday, 13 December.
(Via Blackboard, on paper, or by email as a last resort.)

Instructions: Do both of parts −, and, if you wish, part = as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part −. Do any *eight* (8) of problems **1** – **15**. *[80 = 8 × 10 each]*

1. The Island of Knights and Knaves has only those two kinds of inhabitants. Knights always tell the truth and knaves always lie. You meet nine inhabitants: Bill, Bart, Rex, Joe, Carl, Bozo, Homer, Sue and Dave.

Bill says that it's not the case that Rex is a knave. Bart claims that either Dave is a knight or Bill is a knave. Rex says, "Joe and I are both knights." Joe claims that both Bart is a knave and Rex is a knight. Carl says, "At least one of the following is true: that Joe is a knave or that Bill is a knave." Bozo says that Homer is a knave and Dave is a knight. Homer tells you that it's not the case that Bart is a knave. Sue claims that neither Rex nor Carl are knaves. Dave tells you that at least one of the following is true: that Joe is a knave or that Sue is a knave.

Determine which of the nine is a knight and which is a knave. *[10]*

2. The Fibonacci numbers, f_n for each $n \in \mathbb{N}$, are given by the following recursive definition: $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$. Use induction to show that $f_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$ for all $n \in \mathbb{N}$. *[10]*

3. Suppose $<_A$ is a (strict) linear order on a set A with the property that every non-empty subset $B \subseteq A$ has both a greatest element and a least element in B itself. Show that A must be a finite set. *[10]*

4. Suppose that instead of using schnitts, the real numbers were defined as equivalence classes of Cauchy sequences of rational numbers:

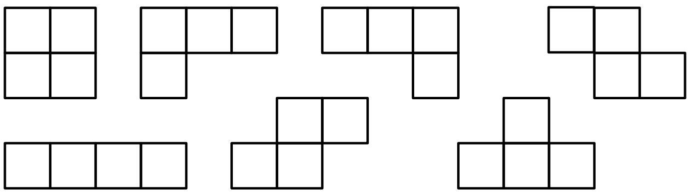
> Recall that a sequence $\{q_n\}$ is *Cauchy* if for every $\epsilon > 0$, there is an N such that for all $m, k \geq N$, $|q_m - q_k| < \varepsilon$. Define an equivalence relation $\approx$ on Cauchy sequences of rationals by $\{q_n\} \approx \{p_n\}$ if for all $\varepsilon > 0$, there is an N such that for all $n \geq N$, $|q_n - p_n| < \varepsilon$. Then $\mathbb{R} = \left\{ [\{q_n\}]_{\approx} \mid \{q_n\} \text{ is a Cauchy sequence of rationals} \right\}$.

Define multiplication on the real numbers using this definition and check that it is commutative. *[10]*

5. A natural number $n > 1$ is said to be *perfect* if it is the sum of its divisors other than itself. (For example, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are the two smallest perfect numbers.) Suppose p and $2^p - 1$ are both prime. Show that $n = 2^{p-1}(2^p - 1)$ is perfect. *[10]*

6. Suppose $P(x)$ is a one-place relation in a first-order language. Write a formula in the language that expresses the statement "There are at least four possible values of x for which $P(x)$ is true." *[10]*

7. *Tetrominoes* are shapes obtained by glueing four 1×1 squares together full edge to full edge. In some cases, such as the game *Tetris*, two tetrominoes that can be made congruent via rotations are considered to be the same, but reflections (*i.e.* flips) are not allowed. This gives seven different tetrominoes:



- **a.** Show how to completely cover a 7×14 rectangle with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 7×14 rectangle, or explain why no such covering can exist. *[5]*
- **b.** Show how to completely cover a 10×10 square with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 10×10 square, or explain why no such covering can exist. *[5]*

8. Show that every natural number n is equal to a sum of the form

$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1! + a_0 \cdot 0!$$

for some $k \geq 0$ and such that each a_i is a natural number with $0 \leq a_i \leq i$. *[10]*

9. Define the logical connective $\uparrow$ via the following truth table:

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

- **a.** Write a formula truth-table equivalent to $A \uparrow B$ using the logical connectives $\neg$ and $\rightarrow$. *[2]*
- **b.** Write formulas truth-table equivalent to $\neg A$, $A \vee B$, $A \wedge B$, and $A \Rightarrow B$ using just the connective $\uparrow$. *[8]*

NOTE: In both parts **a** and **b** you may use the connective(s) you are supposed to use as many times as you like in the desired formula.

10. Show that the set ${}^{\mathbb{N}}2$ of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable. *[10]*

11. Suppose a and r are positive real numbers. Define a sequence a_n, $n \in \mathbb{N}$, of real numbers by setting $a_0 = a$ and, given that a_n has been defined, setting $a_{n+1} = ra_n + a$. Find a closed formula for a_n in terms of a, r, and n. *[10]*

12. Define a binary relation W on the real numbers by $r\mathrm{W}s \iff r = s + 2\pi k$ for some integer k. The *wheel numbers* are the equivalence classes of this relation, *i.e.* $\mathbb{W} = \{\, [r]_\mathrm{W} \mid r \in \mathbb{R} \,\}$. Define addition for wheel numbers by $[r]_\mathrm{W} + [s]_\mathrm{W} = [r + s]_\mathrm{W}$, and let $0_\mathrm{W} = [0]_\mathrm{W}$ and $1_\mathrm{W} = [1]_\mathrm{W}$.

- **a.** Show that addition for the wheel numbers is well-defined. *[2]*
- **b.** Show that addition for the wheel numbers is commutative and associative. *[4]*
- **c.** Can one define a linear order $<_\mathrm{W}$ on $\mathbb{W}$ such that $w <_\mathrm{W} w + 1_\mathrm{W}$ for all $w \in \mathbb{W}$? Either give a definition of such a linear order with an explanation of why it works, or explain why there cannot be such a linear order. *[4]*

13. Suppose that for each $n \in \mathbb{N}$, A_n is an infinite and countable set. Show that the union of all the A_n, $A = \bigcup_{n=0}^{\infty} A_n$, is also countable. *[10]*

14. Prove the right cancellation law for multiplication on the natural numbers, *i.e.* that for all a, b, $c \in \mathbb{N}$ with $c \neq 0$, if $a \cdot_\mathbb{N} c = b \cdot_\mathbb{N} c$, then $a = b$. *[10]*

15. Consider the open interval $(0, 1)$ in the real numbers. Show that $\|(0, 1)\| = \|\mathbb{R}\|$. *[10]*

[Total = 80]

Part =. Bonus questions!

|. Write an original poem about logic or mathematics. *[1]*

||. Why call the number system $\mathbb{W}$ in question **12** the wheel numbers? *[0.5]*

|||. Is $p(n) = n^2 - n + 41$ a prime number for every $n \in \mathbb{N}$? Prove it is or give a counterexample. *[0.5]*

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY YOUR BREAK!