Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Assignment #8
The Quotient Numbers
Due on Friday, 8 November.[*]

Recall our definition of the rational numbers:

- We defined an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{ (a,b) \mid a,\, b \in \mathbb{Z} \text{ and } b \neq 0 \}$ by $(a,b) \sim (c,d) \iff ad = bc$. Informally, $(a,b) \sim (c,d)$ exactly when $\frac{a}{b} = \frac{c}{d}$.
- The equivalence class $[(a,b)]_\sim$ of a pair $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ consists of all the pairs equivalent to (a,b), *i.e.* $[(a,b)]_\sim = \{ (c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \mid (a,b) \sim (c,d) \}$. Informally, $[(a,b)]_\sim$ groups all the pairs (c,d) such that $\frac{a}{b} = \frac{c}{d}$.
- The set of rational numbers is then officially $\mathbb{Q} = \left\{ [(a,b)]_\sim \mid (a,b) \in BbbZ \times (\mathbb{Z} \setminus \{0\}) \right\}$. You can think of $[(a,b)]_\sim$ as being the official "value" of the fraction $\frac{a}{b}$ (and every other fraction equal to it).

Having defined $\mathbb{Q}$, we also defined addition and, multiplication, and the linear order on the rationals as follows.

- Officially, $[(a,b)]_\sim +_{\mathbb{Q}} [(c,d)]_\sim = [(ad+bc, bd)]_\sim$. Informally, this is just the hopefully familiar fact that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.
- Officially, $[(a,b)]_\sim \cdot_{\mathbb{Q}} [(c,d)]_\sim = [(ac, bd)]_\sim$. Informally, this is just the hopefully familiar fact that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- Officially, $[(a,b)]_\sim <_{\mathbb{Q}} [(c,d)]_\sim \iff ad < bc$, where we may assume that both b and d are positive. Informally, this is just the fact that $\frac{a}{b} < \frac{c}{d}$ exactly when cross-multiplying gives us $ad < bc$, which is easy to check as long we are cross-multiplying by positive denominators.

1. Verify that the right distributive law holds in the rationals; that is, if $r,\, s,\, t \in \mathbb{Q}$, then $r \cdot_{\mathbb{Q}} \left(s +_{\mathbb{Q}} t \right) = \left(r \cdot_{\mathbb{Q}} s \right) +_{\mathbb{Q}} \left(r \cdot_{\mathbb{Q}} t \right)$. *[3]*

2. Show that if $r,\, s,\, t \in \mathbb{Q}$ and $r <_{\mathbb{Q}} s$, then $r + t <_{\mathbb{Q}} s + t$. *[4]*

3. Show that the linear order $<_{\mathbb{Q}}$ on the rational numbers has no endpoints; that is, there is no smallest and no largest rational number in this linear order. *[3]*

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[*] Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can.