Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #8
The Quotient Numbers

Recall our definition of the rational numbers:

- We defined an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{\, (a,b) \mid a,\, b \in \mathbb{Z} \text{ and } b \neq 0 \,\}$ by $(a,b) \sim (c,d) \iff ad = bc$. Informally, $(a,b) \sim (c,d)$ exactly when $\frac{a}{b} = \frac{c}{d}$.
- The equivalence class $[(a,b)]_\sim$ of a pair $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ consists of all the pairs equivalent to (a,b), *i.e.* $[(a,b)]_\sim = \{\, (c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \mid (a,b) \sim (c,d) \,\}$. Informally, $[(a,b)]_\sim$ groups all the pairs (c,d) such that $\frac{a}{b} = \frac{c}{d}$.
- The set of rational numbers is then officially $\mathbb{Q} = \left\{\, [(a,b)]_\sim \mid (a,b) \in BbbZ \times (\mathbb{Z} \setminus \{0\}) \,\right\}$. You can think of $[(a,b)]_\sim$ as being the official "value" of the fraction $\frac{a}{b}$ (and every other fraction equal to it).

Having defined $\mathbb{Q}$, we also defined addition and, multiplication, and the linear order on the rationals as follows.

- Officially, $[(a,b)]_\sim +_\mathbb{Q} [(c,d)]_\sim = [(ad+bc, bd)]_\sim$. Informally, this is just the hopefully familiar fact that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.
- Officially, $[(a,b)]_\sim \cdot_\mathbb{Q} [(c,d)]_\sim = [(ac, bd)]_\sim$. Informally, this is just the hopefully familiar fact that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- Officially, $[(a,b)]_\sim <_\mathbb{Q} [(c,d)]_\sim \iff ad < bc$, where we may assume that both b and d are positive. Informally, this is just the fact that $\frac{a}{b} < \frac{c}{d}$ exactly when cross-multiplying gives us $ad < bc$, which is easy to check as long we are cross-multiplying by positive denominators.

1. Verify that the right distributive law holds in the rationals; that is, if r, s, $t \in \mathbb{Q}$, then $r \cdot_\mathbb{Q} \left(s +_\mathbb{Q} t\right) = \left(r \cdot_\mathbb{Q} s\right) +_\mathbb{Q} \left(r \cdot_\mathbb{Q} t\right)$. *[3]*

SOLUTION. Suppose that $r = [(a,b)]_\sim$, $s = [(c,d)]_\sim$, and $t = [(e,f)]_\sim$. Then, using the properties of multiplication and addition on the integers at the key steps, we have:

$$\begin{aligned} r \cdot_\mathbb{Q} \left(s +_\mathbb{Q} t\right) &= [(a,b)]_\sim \cdot_\mathbb{Q} \left([(c,d)]_\sim +_\mathbb{Q} [(e,f)]_\sim\right) = [(a,b)]_\sim \cdot_\mathbb{Q} [(cf+de, df)]_\sim \\ &= [(a(cf+de), bdf)]_\sim = [(a(cf+de), bdf)]_\sim \cdot_\mathbb{Q} 1_\mathbb{Q} \\ &= [(a(cf+de), bdf)]_\sim \cdot_\mathbb{Q} [(b,b)]_\sim = [(a(cf+de)b, bdfb)]_\sim \\ &= [(abcf+abde, bdbf)]_\sim = [(acbf+bdae, bdbf)]_\sim = [(ac,bd)]_\sim +_\mathbb{Q} [(ae,bf)]_\sim \\ &= \left([(a,b)]_\sim \cdot_\mathbb{Q} [(c,d)]_\sim\right) +_\mathbb{Q} \left([(a,b)]_\sim \cdot_\mathbb{Q} [(e,f)]_\sim\right) = \left(r \cdot_\mathbb{Q} s\right) +_\mathbb{Q} \left(r \cdot_\mathbb{Q} t\right) \quad \square \end{aligned}$$

2. Show that if r, s, $t \in \mathbb{Q}$ and $r <_\mathbb{Q} s$, then $r + t <_\mathbb{Q} s + t$. *[4]*

SOLUTION. Suppose that $r = [(a,b)]_\sim$, $s = [(c,d)]_\sim$, and $t = [(e,f)]_\sim$, and that $r <_\mathbb{Q} s$, *i.e.* $ad < bc$. Then, using the properties of multiplication, addition, and the linear order on the integers at the key steps, we have:

$$\begin{aligned} r +_\mathbb{Q} t &= [(a,b)]_\sim +_\mathbb{Q} [(e,f)]_\sim = [(af+be, bf)]_\sim \\ &<_\mathbb{Q} [(cf+de, df)]_\sim = [(c,d)]_\sim +_\mathbb{Q} [(e,f)]_\sim = s +_\mathbb{Q} t \end{aligned}$$

For the key step, note that $ad < bc$ implies that $adf^2 < bcf^2$ (as $f \neq 0$ and so $f^2 > 0$), which in turn implies that $adf^2 + bdef < bcf^2 + bdef$, so $(af+be)df = adf^2 + bdef < cbf^2 + bdef = (cf+de)bf$, which means that $[(af+be, bf)]_\sim <_\mathbb{Q} [(cf+de, df)]_\sim$ be the definition of $<_\mathbb{Q}$. $\square$

3. Show that the linear order $<_{\mathbb{Q}}$ on the rational numbers has no endpoints; that is, there is no smallest and no largest rational number in this linear order. *[3]*

SOLUTION. Suppose that $r = [(a,b)]_\sim \in \mathbb{Q}$, where we may – and do – assume that $b > 0$. We will show that there are both larger and smaller elements of $\mathbb{Q}$ in the linear order $<_{\mathbb{Q}}$. This means, in particular, that r cannot be an endpoint for the linear order. We will show that $r - 1_{\mathbb{Q}} <_{\mathbb{Q}} r <_{\mathbb{Q}} r + 1_{\mathbb{Q}}$, where $1_{\mathbb{Q}} = [(1,1)]_\sim$.

Recall that $r - 1_{\mathbb{Q}} = [(a,b)]_\sim + \left(- [(1,1)]_\sim\right) = [(a,b)]_\sim + [(-1,1)]_\sim = [(a1 + b(-1), b1)]_\sim = [(a-b,b)]_\sim$. Using the fact that $b > 0$, and the properties of multiplication, addition, and the linear order on the integers, it follows that $a - b < a \implies (a-b)b < ab \implies r - 1_{\mathbb{Q}} = [(a-b,b)]_\sim < [(a,b)]_\sim = r$.

Similarly, $r + 1_{\mathbb{Q}} = [(a,b)]_\sim + [(1,1)]_\sim = [(a1 + b1, b1)]_\sim = [(a+b,b)]_\sim$. Using the fact that $b > 0$, and the properties of multiplication, addition, and the linear order on the integers, it follows that $a < a + b \implies ab < (a+b)b \implies r = [(a,b)]_\sim <_{\mathbb{Q}} [(a+b,b)]_\sim = r + 1_{\mathbb{B}}$. ■