Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Assignment #5
Exponentiation on $\mathbb{N}$
Due on Friday, 11 October.[*]

Recall that addition of natural numbers is defined recursively using the successor function as follows:

- For all $n \in \mathbb{N}$, $n + 0 = n$.
- For all n, $k \in \mathbb{N}$, if $n + k$ has been defined, then $n + S(k) = S(n + k)$.

Similarly, multiplication of natural numbers is defined recursively using addition and the successor function as follows:

- For all $n \in \mathbb{N}$, $n \cdot 0 = 0$.
- For all n, $k \in \mathbb{N}$, if $n \cdot k$ has been defined, then $n \cdot S(k) = (n \cdot k) + n$.

In what follows, you may assume that both addition and multiplication of real numbers have all the familiar algebraic properties, including the cancellation and distributive laws.

1. Give a recursive definition of exponentiation of natural numbers. It should satisfy the convention that $0^0 = 1$. *[2]*

2. Use induction to show that for all a, b, $c \in \mathbb{N}$, $\left(a^b\right)^c = a^{b \cdot c}$. *[6]*

3. Show that exponentiation of natural numbers is not always commutative. *[1]*

4. Is exponentiation of natural numbers always associative or not? Prove that it is or give a counterexample. *[1]*

[*] Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can.