Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #5
Exponentiation on ℕ

Recall that addition of natural numbers is defined recursively using the successor function as follows:

- For all $n \in \mathbb{N}$, $n + 0 = n$.
- For all n, $k \in \mathbb{N}$, if $n + k$ has been defined, then $n + S(k) = S(n + k)$.

Similarly, multiplication of natural numbers is defined recursively using addition and the successor function as follows:

- For all $n \in \mathbb{N}$, $n \cdot 0 = 0$.
- For all n, $k \in \mathbb{N}$, if $n \cdot k$ has been defined, then $n \cdot S(k) = (n \cdot k) + n$.

In what follows, you may assume that both addition and multiplication of real numbers have all the familiar algebraic properties, including the cancellation and distributive laws.

1. Give a recursive definition of exponentiation of natural numbers. It should satisfy the convention that $0^0 = 1$. *[2]*

SOLUTION. Here we go; keep in mind that 1 is shorthand for $S(0)$.

- For all $n \in \mathbb{N}$, $n^0 = 1$.
- For all n, $k \in \mathbb{N}$, if n^k has been defined, then $n^{S(k)} = \left(n^k\right) \cdot n$.

Note that $0^0 = 1$ by the first part of the definition and that $0^m = 0$ for all $m > 0$ by the second part of the definition. ■

2. Use induction to show that for all a, b, $c \in \mathbb{N}$, $\left(a^b\right)^c = a^{b \cdot c}$. *[6]*

SOLUTION. We will proceed by induction on c:

Base Step. $(c = 0)$ For all a, $b \in \mathbb{N}$, $\left(a^b\right)^0 = 1 = a^0 = a^{b \cdot 0}$, by the defintion of exponentiation and since $b \cdot 0 = 0$ by the definition of multiplication.

Induction Hypothesis. $(c = k)$ Assume that for all a, $b \in \mathbb{N}$ and some $k \in \ltimes$, $\left(a^b\right)^c k = a^{b \cdot k}$

Inductive Step. $(c = k \to c = S(k))$ For all a, $b \in \mathbb{N}$,

$$\left(a^b\right)^{S(k)} = \left(\left(a^b\right)^k\right) \cdot \left(a^b\right) = \left(a^{b \cdot k}\right) \cdot \left(a^b\right) = a^{(b \cdot k) + b} = a^{b \cdot S(k)},$$

as desired, except for justifying the step $\left(a^{b \cdot k}\right) \cdot \left(a^b\right) = a^{(b \cdot k) + b}$. This requires knowing that $\left(d^e\right) \cdot \left(d^f\right) = d^{e+f}$ for natural numbers d, e, and f. We prove this fact by induction on f:

> *Base Step.* $(f = 0)$ For all d, $e \in \mathbb{N}$, $\left(d^e\right) \cdot \left(d^0\right) = \left(d^e\right) \cdot 1 = d^e = d^{e+0}$ by the definition of exponentiation and the properties of multiplication and addition.
>
> *Induction Hypothesis.* $(f = k)$ Assume that for all d, $e \in \mathbb{N}$ and some $f \in \mathbb{N}$, $\left(d^e\right) \cdot \left(d^f\right) = d^{e+f}$.

Inductive Step. ($f = k \rightarrow f = S(k)$) For all d, $a \in \mathbb{N}$,

$$(d^e) \cdot \left(d^{S(f)}\right) = (d^e) \cdot \left(\left(d^f\right) \cdot d\right) = \left((d^e) \cdot \left(d^f\right)\right) \cdot d = \left(d^{e+f}\right) \cdot d = d^{S(e+f)} = d^{e+S(f)},$$

using assorted properties of addition and multiplication as well as definitions and the Inductive Hypothesis.

Thus, by mathematical induction, $(d^e) \cdot \left(d^f\right) = d^{e+f}$ for all natural numbers d, e, and f. □

Hence, by mathematical induction – and lots of it! – $\left(a^b\right)^c = a^{b \cdot c}$ for all natural numbers a, b, and c. ■

3. Show that exponentiation of natural numbers is not always commutative. *[1]*

Solution. One small counterexample would be $1^2 = 1 \neq 2 = 2^1$.

Of course, if we're paranoid, we really ought to check that $1^2 = 1$ and $2^1 = 2$. Recall that 1 is technically shorthand for $S(0)$ and 2 is technically shorthand for $S(1) = s\left(S(0)\right)$.

$$\begin{aligned}
1^2 &= 1^{S(1)} = 1^1 \cdot 1 = 1^1 = 1^{S(0)} = 1^0 \cdot 1 = 1 \cdot 1 = 1 \cdot S(0) = 1 \cdot 0 + 1 = 0 + 1 = 1 \\
2^1 &= 2^{S(0)} = 2^0 \cdot 2 = 1 \cdot 2 = 1 \cdot S(1) = 1 \cdot 1 + 1 = 1 \cdot S(0) + 1 = (1 \cdot 0 + 1) + 1 \\
&= (0 + 1) + 1 = 1 + 1 = 1 + S(0) = S(1 + 0) = S(1) = 2
\end{aligned}$$

For the truly paranoid, how do we know that $1 \neq 2$? :-) ■

4. Is exponentiation of natural numbers always associative or not? Prove that it is or give a counterexample. *[1]*

Solution. Expoentiation of natural numbers is not associative most of the time. For example, $\left(2^2\right)^3 = 2^6 = 64 \neq 512 = 2^9 = 2^{\left(2^3\right)}$. We'll leave the details for the paranoid to the paranoid . . . :-) ■