Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #4
A Bit of Algebra via Set Theory and Induction

1. Use the Zermelo-Fraenkel axioms of set theory to show that the successor function, $S(x) = x \cup \{x\}$ is 1–1, *i.e.* that if $S(x) = S(y)$, then $x = y$. *[4]*

SOLUTION. Suppose that we have $S(x) = S(y)$ for some sets x and y. By definition, this means that $x \cup \{x\} = S(x) = S(y) = y \cup \{y\}$. It follows that $x \in y \cup \{y\}$, and so $x \in y$ or $x \in \{y\}$.

In the former case, that $x \in y$, we would have to have $x \neq y$ because, as was shown in class, the Axiom of Fondation implies that no set can be an element of itself. However, $y \in x \cup \{x\}$, too, and if $x \neq y$, we would have to have $y \in x$, too. By question **2** on Assignment #3, it isn't possible to have both $x \in y$ and $y \in x$, so the former case cannot be true.

Thus the latter case, that $x \in \{y\}$, must be true, which implies that $x = y$. ■

2. Use induction to show that addition on the natural numbers satisfies the *Right Cancellation Law*: for all a, b, $n \in \mathbb{N}$, if $a + n = b + n$, then $a = b$. *[5]*

Hint: **1** is the key to **2**.

SOLUTION. We will show that $a + n = b + n$ implies that $a = b$, for all a, b, $n \in \mathbb{N}$, by induction on n.

Base Step. ($n = 0$) For all a, $b \in \mathbb{N}$, if $a + 0 = b + 0$, then we have $a = a + 0 = b + 0 = b$ by the definition of addition, so $a = b$.

Inductive Hypothesis. Assume that $a + n = b + n$ implies that $a = b$, for all a, b, $\in \mathbb{N}$ and some $n \in \mathbb{N}$.

Inductive Step. ($n \Rightarrow S(n)$) We need to show that if a, b, $\in \mathbb{N}$ and $a + S(n) = b + S(n)$, then $a = b$. Observe that if a, b, $\in \mathbb{N}$ and $a + S(n) = b + S(n)$, then $S(a+n) = a + S(n) = b + S(n) = S(b+n)$ by the definition of addition. By question **1**, the successor function is 1–1, so $a + n = b + n$, from which it follows that $a = b$ by the Inductive Hypothesis.

Thus, by induction, the addition on the natural numbers satisfies the Right Cancellation Law. ■

3. What can you deduce from the Right Cancellation Law for addition if you also know that addition on the natural numbers is commutative? *[1]*

SOLUTION. If we know that addition on the natural numbers is commutative, *i.e.* $a + b = b + a$ for all a, $b \in \mathbb{N}$, then we can deduce the *Left Cancellation Law* for addition, that for all a, b, $n \in \mathbb{N}$, if $n + a = n + b$, then $a = b$, from the Right Cancellation Law. (The details are left to the reader. :-) ■