Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2024

Solutions to Assignment #3
A Little Set Theory

You should probably check out the axioms described in the handout *The Zermelo-Fraenkel Axioms of Set Theory* before tackling this assignment. Note that these axioms are given somewhat informally – manifestly not in the formal language for set theory mentioned in class – so you should give similarly informal arguments in answering the questions below. Should you try to answer these questions using that language and formal deductions, you will probably regret it ...

1. Suppose x is a set. Give an informal proof using the Zermelo-Fraenkel axioms that the *successor* of x, namely $S(x) = x \cup \{x\}$, is also a set. *[5]*

Solution. By the Pair Set Axiom, if x is a set, then $\{ x, x \}$ is a set; by the Axiom of Extensionality, however, $\{ x, x \} = \{x\}$ because each element of one is an element of the other. Thus $\{x\}$ is a set.

Applying the Pair Set Axiom again, since x and $\{x\}$ are both sets, so is $\{ x, \{x\} \}$. It follows by the Union Axiom that the union of the lements of this set is also a set, *i.e.* $\bigcup \{ x, \{x\} \} = x \cup \{x\} = S(x)$ is a set. ■

2. Suppose u and w are sets. Give an informal proof using the Zermelo-Fraenkel axioms showing that is not possible to have both $u \in w$ and $w \in u$. *[5]*

Hint: The Axiom of Foundation is the key to **2**.

Solution. Suppose, by way of contradiction, that there were indeed sets u and w such that $u \in w$ and $w \in u$. Then $x = \{ u, w \}$ would also be a set by the Pair Set Axiom. Note that we would have $w \in u \cap x$ and $u \in w \cap x$, so neither of $u \cap x$ and $w \cap x$ would be empty. This means that there is no element $y \in x$ such that $y \cap x = \emptyset$, contradicting the Axiom of Foundation.

Since assuming that such sets existed led to a contradiction, there cannot be sets u and w such that $u \in w$ and $w \in u$. ■