Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Solutions to Assignment #2
Propositional Logic

All references are to the pretty minimal system of propositional logic described in class, and also in the handout *A Minimal System of Propositional Logic – The Short Form*.

1. Determine the possible lengths, as sequences of symbols, of (official) formulas of this system. *[4]*

SOLUTION. Recall the formation rules for official formulas of our language for propositional logic:

a. Every atomic formula is a formula.
b. If α is a formula, then $(\neg\alpha)$ is a formula.
c. If α and β are formulas, then $(\alpha \to \beta)$ is a formula.
d. A string of symbols of $\mathcal{L}_P$ is a formula only if it is obtainedcan be built from the symbols of $\mathcal{L}_P$ by finitely many applications of rules a, b, and c.

If these rules apply, every integer length $n \geq 1$ is possible, except for $n = 2$, $n = 3$, and $n = 6$.

$n = 1$. Every atomic formula is a single symbol, so there are formulas of length 1.

$n = 2$. The only way to make formulas longer than atomic formulas is to either negate an existing formula, which adds 3 symbols to an existing total of at least 1 symbol for a total of at least 4, or to make an implication between two existing formulas, which adds 3 symbols to an existing total of at least 2 (*i.e.* at least 1 for each existing formula) for a total of at least 5. Thus no formula can have only two symbols.

$n = 3$. Ditto, except that no formula can have only three symbols.

$n = 4$. The negation of any atomic formula, *e.g.* $(\neg A_7)$, has 4 symbols.

$n = 5$. An implication between any two atomic formulas, *e.g.* $(A_3 \to A_{16})$, has 5 symbols.

$n = 6$. Suppose φ was a formula with exactly 6 symbols. It would then have to be either a negation $(\neg\beta)$ of some formula β that had three symbols, which is impossible, or an implication $(\gamma \to \delta)$ between two formulas γ and δ which had three symbols between them, meaning that one of them had two symbols, which is impossible. Thus there can be no formula with exactly 6 symbols.

$n = 7$. Here is an example of a formula with 7 symbols: $(\neg\,(\neg A_7))$.

$n = 8$. Here is an example of a formula with 8 symbols: $(\neg\,(A_3 \to A_{16}))$.

$n = 9$. Here is an example of a formula with 9 symbols: $(A_5 \to (A_3 \to A_{16}))$.

$\vdots$

$n = 3k + 1$. $(k > 2)$ Negate a formula with 7 symbols $k - 2$ times to get a formula with $3(k-2) + 7 = 3k + 1$ symbols.

$n = 3k + 2$. $(k > 2)$ Negate a formula with 8 symbols $k - 2$ times to get a formula with $3(k-2) + 8 = 3k + 2$ symbols.

$n = 3k + 3$. $(k > 2)$ Negate a formula with 9 symbols $k - 2$ times to get a formula with $3(k-2) + 9 = 3k + 3$ symbols.

It follows that official formulas of our system of proposition logic can have any integer length $n \geq 1$, except for $n = 2$, $n = 3$, and $n = 6$. ■

2. Use a truth table to verify that whenever the formulas $(\alpha \to (\beta \to \gamma))$ and β are both true, then the formula $(\alpha \to \gamma)$ is also true. *[2]*

SOLUTION. Here is the truth table:

α	β	γ	$(\alpha \to (\beta \to \gamma))$	$(\alpha \to \gamma)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Note $(\alpha \to \gamma)$ is true in every line of the truth table in which both $(\alpha \to (\beta \to \gamma))$ and β are true, as desired. ■

3. Use a deduction to verify that whenever the formulas $(\alpha \to (\beta \to \gamma))$ and β are both true, then the formula $(\alpha \to \gamma)$ is also true. *[4]*

SOLUTION. Here is a deduction using the premisses $(\alpha \to (\beta \to \gamma))$ and β which has $(\alpha \to \gamma)$ as its conclusion.

1.	$((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$	A2
2.	$(\alpha \to (\beta \to \gamma))$	Premiss
3.	$((\alpha \to \beta) \to (\alpha \to \gamma))$	1,2 MP
4.	$(\beta \to (\alpha \to \beta))$	A1
5.	β	Premiss
6.	$(\alpha \to \beta)$	4,5 MP
7.	$(\alpha \to \gamma)$	3,6 MP

That's that! ■