Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2024

Solution to Assignment #11
Open or Closed?

Here are two definitions of *open* and *closed* for subsets of $\mathbb{R}$:

DEFINITION 1. $A \subseteq \mathbb{R}$ us said to be *open* if for every $a \in A$ there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq A$. $B \subseteq \mathbb{R}$ is said to be *closed* if $\mathbb{R} \setminus B$ is open.

DEFINITION 2. $B \subseteq \mathbb{R}$ is *closed* if for every sequence $\{b_n\}$ such that each $b_n \in B$ that has a limit $\lim_{n\to\infty} b_n = b$, we also have $b \in B$. $A \subseteq \mathbb{R}$ is said to be *open* if $\mathbb{R} \setminus A$ is closed.

The first definition is more common, with the second then becoming a theorem that usually gets proved fairly early because of its utility.

1. Show that the two definitions of open and closed for subsets of $\mathbb{R}$ are equivalent. *[10]*

NOTE. Not every subset of $\mathbb{R}$ is open or closed. For a cheap example, an interval such as $(0, 1]$ which contains one of its endpoints, but not the other, is neither open nor closed.

SOLUTION. First, suppose that $B \subseteq \mathbb{R}$ is closed in the sense of Definition 1. This means that $A = \mathbb{R} \setminus B$ is open in the sense of Definition 1, *i.e.* for every $a \in A$ there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq A$. We will show that B is closed in the sense of Definition 2.

Suppose $\{b_n\}$ is sequence of elements of B which has a limit $b = \lim_{n\to\infty} b_n$. We need to show that $b \in B$ as well. Assume, by way of contradiction, that $b \notin B$. Then $b \in A = \mathbb{R} \setminus B$, so there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq A = \mathbb{R} \setminus B$. Since each $b_n \in B$, it follows that $|b_n - b| \geq \varepsilon$ for all n, which contradicts $b = \lim_{n\to\infty} b_n$. Thus B must be closed in the sense of Definition 2.

Note that because in both definitions open sets are the complements of closed sets, the argument above also implies that sets open in the sense of Definition 1 are open in the sense of Definition 2.

Second, suppose $A \subseteq \mathbb{R}$ is open in the sense of Definition 2. This means that $B = \mathbb{R} \setminus A$ is closed in the sense of Definition 2, *i.e.* for every sequence $\{b_n\}$ such that each $b_n \in B$ that has a limit $\lim_{n\to\infty} b_n = b$, we also have $b \in B$. We will show that A is open in the sense of Definition 1.

Suppose $a \in A$. Let $\varepsilon = \inf \{\, |b - a| \mid b \in B = \mathbb{R} \setminus A \,\}$.