**Mathematics 2200H – Mathematical Reasoning**
TRENT UNIVERSITY, Fall 2024

**Solution to Assignment #10**
**Monotonic Subsequences**

Recall from class that a sequence $\{a_n\}$ of real numbers is said to be *monotonic* if it is non-increasing, *i.e.* $a_n \geq a_{n+1}$ for all $n$, or non-decreasing, *i.e.* $a_n \leq a_{n+1}$ for all $n$. A *subsequence* $\{a_{n_i}\}$ of a sequence $\{a_n\}$ is the sequence whose elements are the elements of the sequence $\{a_n\}$ selected by a strictly increasing sequence of indices $n_i$. That is, if $0 \leq n_0 < n_1 < n_2 < \cdots$, the subsequence is $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$. Note that going from $n_i$ to $n_{i+1}$ could skip over a lot of natural numbers.

**1.** Prove that every sequence of real numbers has a monotonic subsequence. *[10]*

SOLUTION. Suppose $\{a_n\}$ is a sequence fof real numbers. We will find a monotonic subsequence of this sequence in two stages.

First, we will select a subsequence $\{b_k\}$ of $\{a_n\}$, where $b_k = a_{n_k}$ for each $k \in \mathbb{N}$, recursively as follows:

$k = 0$. Let $b_0 = a_0$, so $n_0 = 0$. Let
$U_0 = \{\, n \mid n > 0 \;\&\; b_0 = a_0 \leq a_n \,\}$ and
$L_0 = \{\, n \mid n > 0 \;\&\; b_0 = a_0 \geq a_n \,\}$.
At least one of these possibly overlapping sets must be infinite, since they include all the infinitely many elements of $\{\, n \mid n > 0 \,\}$ between them.
Let $M_0 = \begin{cases} U_0 & \text{if } U_0 \text{ is infinite} \\ L_0 & \text{otherwise} \end{cases}$. Note that $n_0 = 0 \notin M_0$.

$k + 1$. Suppose $b_k = a_{n_k}$ and an infinite set $M_k$, with $n_k \notin M_k$, have been defined. Let $b_{k+1} = a_{n_{k+1}}$, where $n_{k+1}$ is the least element of $M_k$. Let
$U_{k+1} = \{\, n \mid n \in M_k \setminus \{n_{k+1}\} \;\&\; b_k = a_{n_k} \leq a_n \,\}$ and
$L_{k+1} = \{\, n \mid n \in M_k \setminus \{n_{k+1}\} \;\&\; b_k = a_{n_k} \geq a_n \,\}$.
At least one of these possibly overlapping sets must be infinite, since they include all the infinitely many elements of $M_k \setminus \{n_{k+1}\}$ between them.
Let $M_{k+1} = \begin{cases} U_{k+1} & \text{if } U_{k+1} \text{ is infinite} \\ L_{k+1} & \text{otherwise} \end{cases}$. Note that $n_{k+1} \notin M_{k+1}$.

Since taking away a single element of an infinite set leaves an infinite set and then dividing that infinite set into two possibly overlapping pieces means that at least one of the pieces is infinite, the process does not terminate at any finite stage. It follows that $\{b_k\}$ is an infinite subsequence of $\{a_n\}$.

Second, we will select a subsequence $\{c_i\}$ of $\{b_k\}$, where $c_i = b_{k_i} = a_{n_{k_i}}$ for each $i \in \mathbb{N}$, as follows. Looking back to the selection of the subsequence $\{b_k\}$ of $\{a_n\}$, let $S = \{\, k \in \mathbb{N} \mid M_k = U_K \,\}$. THen at least one of $S$ and $\mathbb{N} \setminus S$ must be infinite, since between them they have all the elements of the infinite set $\mathbb{N}$.

If $S$ is infinite, let $k_i$, for $i \in \mathbb{N}$, be an enumeration, in order of size, of $S$, and let $c_i = b_{k_i}$. By the definition of the subsequence $\{b_k\}$ and of $S$, we have $c_i = b_{k_i} = a_{n_{k_i}} \leq c_{i+1} = b_{k_{i+1}} = a_{n_{k_{i+1}}}$ for all $i$. Thus, in the case that $S$ infinite, $\{c_i\}$ is a subsequence of $\{a_n\}$ which is non-decreasing.

On the other hand, if $\mathbb{N} \setminus S$ is infinite, let $k_i$, for $i \in \mathbb{N}$, be an enumeration, in order of size, of $\mathbb{N} \setminus S$, and let $c_i = b_{k_i}$. By the definition of the subsequence $\{b_k\}$ and of $S$, we have $c_i = b_{k_i} = a_{n_{k_i}} \geq c_{i+1} = b_{k_{i+1}} = a_{n_{k_{i+1}}}$ for all $i$. Thus, in the case that $\mathbb{N} \setminus S$ infinite, $\{c_i\}$ is a subsequence of $\{a_n\}$ which is non-increasing.

Either way, $\{c_i\}$ is a monotonic subsequence of the original sequence $\{a_n\}$. $\blacksquare$

NOTE. This solution is pretty precise at the cost of being pretty verbose, which may make it harder to actually follow and understand ...