Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Final Examination

Due on Friday, 15 December.
(Via Blackboard, on paper, or by email as a last resort.)

Instructions: Do both of parts $\in$ and $\subseteq$, and, if you wish, part $\equiv$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part $\in$. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. This little puzzle is due to Charles Lutwidge Dodgson, better known by his pen name of Lewis Carroll.

> Five beggars sat down in a circle, and each piled up, in a heap before him, the pennies he had received that day: and the five heaps were equal.
>
> Then spake the eldest and wisest of them, unfolding, as he spake, an empty sack.
>
> "My friends, let me teach you a pretty little game! First, I name myself 'Number One', my left-hand neighbour 'Number Two,' and so on to 'Number Five.' I then pour into this sack the whole of my earnings for the day, and hand it on to him who sits next but one on my left, that is 'Number Three.' *His* part in the game is to take out of it and give to his two neighbours, so many pennies as represent their names (that is, he must give four to 'Number Four' and two to 'Number Two'); he must then put *into* the sack half as much as it ccontained when he received it; and he must then hand it on just as I did, that is, he must hand it to him who sits next but one on his left—who will of course be 'Number Five.' *He* must proceed in the same way, and hand it on to 'Number Two,' from whom the sack will find its way to 'Number Four,' and so to me again. If any player cannot furnish, from his own heap, the whole of what he has to put into the sack, he is at liberty to draw upon any of the other heaps, *except mine*!"
>
> The other beggars entered into the game with much enthusiasm: and in due time the sack returned to 'Number One,' who put into it the two pennies he had received during the game, and carefully tied up the mouth of it with a string. Then, remarking "it is a *very* pretty little game," he rose to his feet, and hastily quitted the spot. The other four beggars gazed at each other with rueful countenances. Not one of them had a penny left!

How much did each beggar have at the beginning? *[10]*

2. Show that $\sqrt{208}$ is irrational. *[10]*

3. Suppose A is a schnitt representing a positive real number a. Use A to define the schnitt representing the real number $\frac{1}{a}$. *[10]*

4. Suppose n is a positive integer with decimal representation $d_k d_{k-1} \cdots d_1 d_0$, *i.e.* $n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10 + d_0$. Show that n is divisible by 3 if and only if the sum of its digits, $d_k + d_{k-1} + \cdots d_1 + d_0$, is divisible by 3. *[10]*

Parts $\subseteq$ and $\equiv$ are on page 2.

See? I told you!

Part $\subseteq$. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Show that $\displaystyle\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$. *[10]*

6. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet nine inhabitants: Zippy, Betty, Bill, Homer, Dave, Zed, Bob, Marge and Carl. Zippy says that Bill is a knave. Betty says that Bill could say that Carl is a knight. Bill tells you that Homer and Marge are different. Homer claims, "Either Bob is a knave or Carl is a knave." Dave says that Betty and Zed are knaves. Zed claims that either Bill is a knave or Dave is a knave. Bob says, "Marge could say that I am a knave." Marge claims, "I am a knight or Zippy is a knight." Carl tells you, "Either Zed is a knave or Bob is a knight."

Determine, as best you can, which of the nine are knights and which are knaves. *[10]*

7. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = \cos(x) + i\sin(x)$.

- **a.** What is the range of $f(x)$, *i.e.* $\{ f(x) \mid x \in \mathbb{R} \}$, as a geometrical object? *[2]*
- **b.** Show that for all $x, y \in \mathbb{R}$, $f(x+y) = f(x) \cdot f(y)$. *[8]*

8. Suppose that for each $n \geq 0$, A_n is a countable set, and that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Show that $A = \displaystyle\bigcup_{n=0}^{\infty} A_n$ is also countable. *[10]*

9. There are 16 possible binary connectives for propositional logic.

- **a.** Give the truth table for each on of the 16 binary connectives. (If you don't know a symbol for a given connective, just invent one. :-) *[6]*
- **b.** Choose any 4 of the 16 binary connectives. For each of your chosen connectives, give a formula using only the connectives $\neg$ and $\to$ that is equivalent to it. *[4]*

10. Show that the set ${}^{\mathbb{N}}2$ of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable. *[10]*

11. Suppose a and r are positive real numbers. Define a sequence a_n, $n \in \mathbb{N}$, of real numbers by setting $a_0 = a$ and, given that a_n has been defined, setting $a_{n+1} = ra_n + a$. Find a closed formula for a_n in terms of a, r, and n. *[10]*

[Total = 80]

Part $\equiv$. Bonus!

$\sim$. Write an original poem about logic or mathematics. *[1]*

$\approx$. Suppose $n \geq 2$ is not prime. Could $3^n - 2$ be prime? *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. ENJOY THE BREAK!