Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Assignment #9
What is the linear order on the rationals missing?
*Due on Friday, 17 November.**

Recall that we defined the rationals by setting $\mathbb{Q} = \left\{ [(a,b)]_\sim \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$, where $\sim$ is the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{ (a,b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \}$ which is defined by $(a,b) \sim (c,d) \iff ad = bc$. Intuitively, the equivalence class $[(a,b)]_\sim$ is the official version of the rational number described by the ratio $\frac{a}{b}$. We then defned the common arithmetic operations on the rationals as follows:

$\cdot$. $[(a,b)]_\sim \cdot [(c,d)]_\sim = [(ac,bd)]_\sim$, *i.e.* $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

$/$. $[(a,b)]_\sim \,/\, [(c,d)]_\sim = [(ad,bc)]_\sim$, *i.e.* $\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

$+$. $[(a,b)]_\sim + [(c,d)]_\sim = [(ad+bc,bd)]_\sim$, *i.e.* $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

$-$. $[(a,b)]_\sim - [(c,d)]_\sim = [(ad+bc,bd)]_\sim$, *i.e.* $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$.

With these definitions it is not hard to check that the identity elements for $+$ and $\cdot$ in the rationals are $0 = [(0,1)]_\sim$ and $1 = [(1,1)]_\sim$, respectively, and that these two operations are associative and commutative and satisfy the distributive laws. It is tedious, but not hard, to check that all of these operations are well-defined in that they don't depend on the particular choices of representatives from the equivalence classes involved.

We also defined the usual linear order on the rationals as follows:

$<$. $[(a,b)]_\sim < [(c,d)]_\sim \iff ad < bc$, *i.e.* $\frac{a}{b} = \frac{c}{d} \iff ad < bc$, provided that pick representatives from each equivalence class with $b > 0$ and $d > 0$. (Why can we?)

Again, it is tedious, but not too hard, to show that the relation $<$ is well-defined, is actually a linear order, and has the properties of having no endpoints and being *dense-in-itself* (or just *dense*), *i.e.* in this linear order there is a rational number between any two different rational numbers.

A linear order $(L, <_L)$ is said to be *complete* if every non-empty set $S \subset L$ that has an *upper bound* in the linear order, *i.e.* for which there is some $u \in L$ such that $s <_L u$ for all $s \in S$, has a *least upper bound* or *supremum* in the linear order, *i.e.* there is some $w \in L$ such that w is an upper bound for S and $w \leq u$ for every upper bound u in S.

1. Show that the usual linear order on the rational numbers is not complete. *[10]*

Hint: You need to find or manufacture a non-empty set of rationals which has an upper bound in the rationals but does not have a least upper bound in the rational numbers. Informally, it may help to think of the rationals as being embedded in the real numbers. (Which last we haven't officially defined yet.)

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* Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.