Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Assignment #5
Induction In Various Forms
Due on Friday, 13 October.[*]

The *Peano Axioms*[†] for "arithmetic", *i.e.* the natural numbers $\mathbb{N}$, are, semi-formally:

1. $0 \in \mathbb{N}$.
2. $\forall n \in \mathbb{N} \, (\, S(n) \in \mathbb{N} \,)$
3. $\forall n \in \mathbb{N} \, (\, S(n) \neq 0 \,)$
4. $\forall n \in \mathbb{N} \, (\, \forall k \in \mathbb{N} \, (\, S(n) = S(k) \leftrightarrow n = k \,) \,)$
5. $\forall X \subseteq \mathbb{N} \, (\, (\, 0 \in X \land \forall x \in X \, (\, S(x) \in X \,) \,) \rightarrow X = \mathbb{N} \,)$

The symbols 0 and S represent the famous constant – it's got books written about it! – and the successor function, respectively. With some work, especially for Peano Axiom 5, one can show that the von Neumann construction of the natural numbers in set theory gives a structure satisfying the Peano Axioms. Every structure that does satisfy the Peano Axioms is, as far as first-order logic can tell, an identical twin of whatever we believe the "real" natural numbers to be. Incidentally, it is in principle possible to use the Peano Axioms and first-order logic as a foundation for all of mathematics, but it would be very cumbersome indeed to do so.

1. Using nothing but the Peano Axioms – so without reference to the von Neumann construction of the natural numbers – define the usual linear order $<$ on the natural numbers. (You need not prove that it is a linear order.) *[3]*

2. Using nothing but the Peano Axioms – so without reference to the von Neumann construction of the natural numbers or the somewhat informal argument given in the solutions to Assignment #2 – show that the natural numbers satisfy the Descending Chain Condition. *[4]*

NOTE. Recall from Assignment #2 that the Descending Chain Condition is the statement that every strictly decreasing sequence of natural numbers is finite.

3. Explain why any structure satisfying the Peano Axioms can be used as a basis for mathematical induction. *[3]*

---

[*] Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.

[†] Named after Giuseppe Peano (1858-1932) , one of the major contributors to the development of mathematical logic and set theory, among other areas of mathematics and linguistics. One of his more interesting results was the construction of a space-filling curve: a function $f : [0,1] \to [0,1] \times [0,1]$ that is continuous and onto. Not everyone who did this stuff was from Central or Eastern Europe. :-)