Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2023

Assignment #3
Propositional Formulas
*Due on Friday, 29 September.**

This assignment is mostly about playing with formulas in propositional logic. Recall from class that our "official" language of propositional logic, $\mathcal{L}_P$, has the following symbols:

i. the *atomic formulas* A_0, A_1, A_2, … ;
ii. the *connectives* $\neg$ and $\rightarrow$; and
iii. the *grouping symbols* (and).

The *formulas* of $\mathcal{L}_P$ are then defined as follows:

a. Every atomic formula is a formula.
b. If α is a formula, then $(\neg\alpha)$ is a formula.
c. If α and β are formulas, then $(\alpha \rightarrow \beta)$ is a formula.
d. No other string of symbols of $\mathcal{L}_P$ is a formula.

The *logical axiom schema* of the system are:

*A*1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$
*A*2. $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$
*A*3. $(((\neg\beta) \rightarrow (\neg\alpha)) \rightarrow (((\neg\beta) \rightarrow \alpha) \rightarrow \beta))$

Every instance of one these schema, where we plug some particular formulas of $\mathcal{L}_P$ in for α, β, and/or γ, is a *logical axiom*.

The system has a single rule of procedure, *Modus Ponens*:

MP. Given the formulas α and $(\alpha \rightarrow \beta)$, we may infer the formula β.

Given a set, possibly empty, of formulas Σ, the (non-logical) *premisses* or *hypotheses*, a *deduction* of a formula α from Σ, is a sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that each formula φ_k in the sequence

1. is a logical axiom,
2. is in Σ (*i.e.* it is a premiss or hypothesis), or
3. follows from some formulas φ_i and φ_j earlier in the sequence (so i, $j < n$),

and φ_n, the last formula in the sequence, is α. $\Sigma \vdash \alpha$, read as "Sigma proves alpha", means that there is a deduction of α from Σ.

1. What are the possible lengths (*i.e.* the number of symbols in the formula, counting repetitions) of formulas of $\mathcal{L}_P$? *[3]*

2. Use truth tables to verify that each of the logical axioms is a *tautology*, *i.e.* a formula that is always true. *[3]*

3. Suppose Σ is some set of hypotheses which are true and that $\Sigma \vdash \alpha$. Show that α must be true. (You'll need to do induction on the length of the deduction here.) *[3]*

4. What are the possible lengths (*i.e.* number of formulas) of deductions in this system? *[1]*

* Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.