Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Assignment #2
Climbing Down
Due on Friday, 22 September.[*]

This assignment is mostly about playing with linear orders, which we discussed a bit in class. Here an equivalent but slightly different version of the definition. A (strict) *linear order*, on a set A is a binary relation $\lhd$ on A which is:

i. *irreflexive*, so $a \not\lhd a$ for all $a \in A$;
ii. *antisymmetric*, so $a \lhd b \iff b \not\lhd a$ for all $a, b \in A$ with $a \neq b$;
iii. *transitive*, so $a \lhd b$ and $b \lhd c$ imply that $a \lhd c$ for all $a, b, c \in A$; and
iv. satisfies *trichotomy*, so for all $a, b \in A$ exactly one of the alternatives $a \lhd b$, $a = b$, or $b \lhd a$ is true.

1. Explain why a binary relation which is transitive and satisfies the version of trichotomy given above must also be irreflexive and antisymmetric. *[3]*

NOTE. **1** means that the definition of linear order given above is redundant in the sense that it only needs the last two conditions. This saves time if you want to check whether some binary relation is a linear order.

The rest of this assignment is less general and is based upon the usual linear order $<$ on the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. (You need not show that this is indeed a linear order. Just take it for granted! :-) One of the interesting extra properties of this linear order is that it satisfies the *Descending Chain Condition*: every strictly decreasing sequence of natural numbers is finite. (A linear order that satisfies this condition is called a *well-order*.)

2. Explain, as best you can, why the usual linear order on the natural numbers satisfies the Descending Chain Condition. *[2]*

The *lexicographic order*[†] on the set $\mathbb{N} \times \mathbb{N} = \{ (n, k) \mid n, k \in \mathbb{N} \}$ of all ordered pairs of natural numbers, which we will denote by $<_L$, is defined as follows: $(a, b) <_L (c, d)$ if and only if $a < c$ or both $a = c$ and $b < d$.

3. Explain why $<_L$ is indeed a linear order on $\mathbb{N} \times \mathbb{N}$. *[3]*

4. Explain why $<_L$ satisfies the descending chain condition. *[2]*

NOTE. Please keep the four Cs of writing mathematics in mind. An argument or explanation should ideally be, in order of priority:

1. Correct
2. Complete
3. Clear
4. Concise

Please give your complete reasoning in your solutions. Don't forget that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

---

[*] Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.

[†] Sometimes called the *dictionary order*.