Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Optional Assignment #12
Ordinal Addition
*Due on Friday, 8 December.**

Recall from class that an *ordinal* is a set α such that

1. $\in$ well-orders α, and
2. if $b \in a \in \alpha$, then $b \in \alpha$.

With this definition, we have that:

i. every natural number (as we defined them in this course) is an ordinal;
ii. the set of natural numbers $\mathbb{N}$ is itself an ordinal, usually denoted as ω when used as an ordinal;
iii. if α is any ordinal, then the successor of α, $S(\alpha) = \alpha \cup \{\alpha\}$, is also an ordinal;
iv. and, for any set of ordinals A, their union, $\bigcup A = \bigcup_{\alpha \in A} \alpha$, is also an ordinal.

An ordinal that is neither 0 nor a successor is said to be a *limit ordinal*. $\omega = \mathbb{N}$ is the smallest limit ordinal.

One of the uses of ordinals is as a framework for recursion and induction "to infinity and beyond". One of the uses of such recursion is the following definition of addition of ordinal numbers:

Zero. If α is any ordinal, then $\alpha + 0 = \alpha$.
Next. If α and β are ordinals and $\alpha + \beta$ has been defined, then $\alpha + S(\beta) = S(\alpha + \beta)$.
Leap. If α is an ordinal and $\alpha + \beta$ has been defined for every ordinal $\beta \in \lambda$, where λ is a limit ordinal, then $\alpha + \lambda = \bigcup_{\beta \in \lambda} (\alpha + \beta)$.

For finite ordinals, *i.e,* natural numbers, this amounts to the usual definition of addition of natural numbers, since limit ordinals don't enter the picture. Once they do, however, addition does not have all of the familiar properties it has in the natural numbers.

1. Show that right cancellation may fail for ordinal addition. That is, find ordinals α, β, and γ such that $\alpha + \gamma = \beta + \gamma$, but $\alpha \neq \beta$. *[4]*

2. Show that left cancellation does work for ordinal addition. That is, show that for all ordinals α, β, and γ, if $\gamma + \alpha = \gamma + \beta$, then $\alpha = \beta$. *[4]*

3. Show that ordinal addition is not always commutative. *[2]*

* Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.