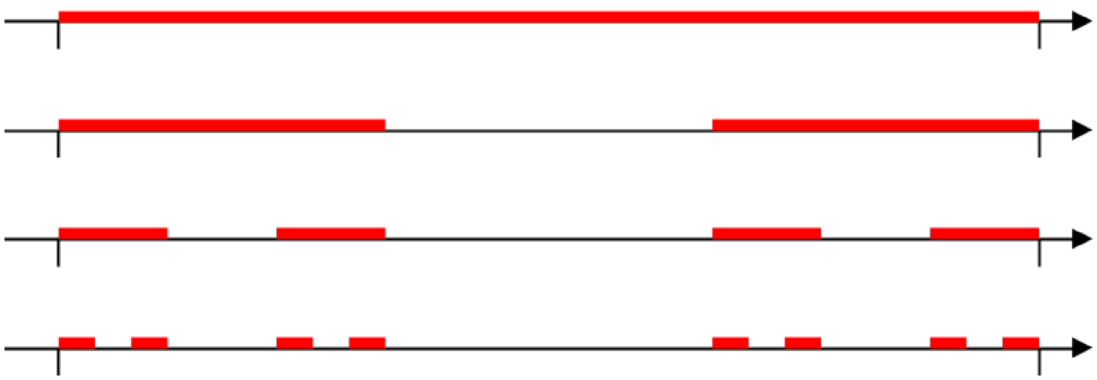
Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2023

Assignment #11
The Cantor Set
Due on Friday, 1 December.[*]

Consider the following process. Start with the unit interval $C_0 = [0.1]$ of real numbers. In step 1, delete the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. In step 2, delete the open middle third of each remaining interval in C_1, leaving $C_2 = \left[1, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. In step 3, delete the open middle third of each remaining open interval in C_2 to make C_3, and so on.



The *Cantor set* $\mathcal{C}$ is what you have left at the end of this process; technically, $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = \{\, x \in [0.1] \mid x \in C_n \text{ for all } n \geq 0 \,\}$. Note that this is not an empty set: the endpoints of every interval comprising C_n for each $n \geq 0$ survive through the reast of the process.

1. What is the total length of the intervals comprising C_n for each $n \geq 0$? *[2]*

The "total length" or *measure* of the Cantor set is not as simple a concept as a sum of lengths of some finite number intervals, not least because $\mathcal{C}$ has no subsets that are intervals of real numbers larger than a single point. However, it is still the limit of the total lengths of the sets C_n as $n \to \infty$.

2. Use the ε–δ definition of the limit of a sequence to verify that the "total length" of the Cantor set $\mathcal{C}$ is 0. *[2]*

3. Show that the interval $[0, 1]$ of real numbers has the same cardinality as the Cantor set $\mathcal{C}$. *[3]*

4. Show that the interval $[0, 1]$ of real numbers has the same cardinality as the set $\mathbb{R}$ of all real numbers. *[3]*

---

[*] Please submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission on Blackboard fails, please submit your solutions to the instructor on paper or via email to `sbilaniuk@ trentu.ca` as soon as you can.