Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022
Solutions to Assignment #9 – (Non-)Completeness

Please give your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

As in Assignment #8, suppose $\mathbb{Q}$ and the usual linear order on the rationals (usually denoted by $<$, or by $<_{\mathbb{Q}}$ when you have other linear orders to keep track of) are defined as they were in class:

- Define the equivalence relation $\approx$ on pairs of integers by
  $(a, b) \approx (c, d)$ if and only if $ad = bc$.
- If (a, b) is a pair of integers with $b \neq 0$, let
  $[(a, b)]_{\approx} = \{ (c, d) \in \mathbb{Z} \times \mathbb{Z} \mid (a, b) \approx (c, d) \}$
  Intuitively, this equivalence class represents the fraction $\frac{a}{b}$.
- Let $\mathbb{Q} = \left\{ [(a, b)]_{\approx} \mid a,\ b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$.
- We can assume that the second coordinate in the pair defining an equivalence class is positive (intuitively, we can assume that the denominator of any fraction is positive). If we stick to such pairs as representatives of equivalence classes, then $[(a, b)]_{\approx} <_{\mathbb{Q}} [(c, d)]_{\approx}$ if and only if $ad <_{\mathbb{Z}} bc$.

To all this we add the following idea: a linear order is *complete* if every non-empty subset that has an upper bound (which need not be in the subset) has a least upper bound (which also need not be in the subset).

In answering the questions below, you may assume that all the familiar properties of the integers, as well as the operations and linear order on the integers, are true.

1. Using these definitions of $\mathbb{Q}$ and $<$, show that $\mathbb{Q}$ is not complete, *i.e.* show that there is a non-empty subset $A \subset \mathbb{Q}$ with an upper bound in $\mathbb{Q}$, but no least upper bound in $\mathbb{Q}$. *[10]*

NOTE. Some non-empty subsets of $\mathbb{Q}$ do have least upper bounds. For example, $\left\{ q \in \mathbb{Q} \mid q < \frac{5}{3} \right\}$ has least upper bound $\frac{5}{3} = [(5, 3)]_{\approx}$.

Hint: Informally, think of the rationals as a part of the real number line.

SOLUTION. Let A be the set of rationals between 0 and $< \sqrt{2}$. That is, unwinding the definitions until we reach formality and then simplify the algebra:

$$\begin{aligned}
A &= \left\{\, q \in \mathbb{Q} \mid 0 < q < \sqrt{2} \,\right\} = \left\{\, q \in \mathbb{Q} \mid 0 < q \text{ and } q^2 < 2 \,\right\} \\
&= \{\, [(a,b)]_\approx \in \mathbb{Q} \mid [(0,1)]_\approx <_\mathbb{Q} [(a,b)]_\approx \\
&\qquad\qquad\qquad\qquad \text{and } [(a,b)]_\approx \cdot_\mathbb{Q} [(a,b)]_\approx <_\mathbb{Q} [(2,1)]_\approx \,\} \\
&= \{\, [(a,b)]_\approx \in \mathbb{Q} \mid [(0,1)]_\approx <_\mathbb{Q} [(a,b)]_\approx \text{ and } \left[\left(a^2,b^2\right)\right]_\approx <_\mathbb{Q} [(2,1)]_\approx \,\} \\
&= \{\, [(a,b)]_\approx \in \mathbb{Q} \mid 0 <_\mathbb{Z} a \text{ and } 0 <_\mathbb{Z} b \text{ and } a^2 \cdot_\mathbb{Z} 1 <_\mathbb{Z} b^2 \cdot_\mathbb{Z} 2 \,\}
\end{aligned}$$

Informally, $A = \left\{\, \frac{a}{b} \mid a \text{ and } b \text{ are positive integers and } a^2 < 2b^2 \,\right\}$.

We claim that A has an upper bound in $\mathbb{Q}$, but has no least upper bound in $\mathbb{Q}$. For the sake of keeping it readable, we'll show these facts a little informally, rather than writing out things in terms of equivalence classes and the like.

First, we claim that $2 = \frac{2}{1}$ is an upper bound for A, *i.e.* $\frac{a}{b} < 2$ for all $\frac{a}{b} \in A$. Suppose, by way of contradiction, that $2 \leq \frac{a}{b}$ for some $\frac{a}{b} \in A$. Note that this means that we can also assume that $a \geq b \geq 1$. Then

$$\begin{aligned}
2 \leq \frac{a}{b} &\implies 2b \leq a \implies 4b^2 \leq a^2 \text{ (since } a \geq b \geq 1) \\
&\implies 4b^2 \leq a^2 < 2b^2 \text{ (since } \tfrac{a}{b} \in A) \\
&\implies 4b^2 < 2b^2 \implies 2 < 1 \text{ (after cancelling } 2b^2 \text{ on both sides)},
\end{aligned}$$

which contradicts the fact that $1 < 2$ in the integers. Thus 2 is an upper bound for A.

Second, we claim that A has no least upper bound. (In the rationals, anyway; as a set of real numbers, it has the least upper bound $\sqrt{2}$.) Since $\sqrt{2}$ is irrational, *i.e.* $\sqrt{2} \neq \frac{a}{b}$ for any rational $\frac{a}{b} \in \mathbb{Q}$, we don't have to worry about the possibility that a least upper bound $\frac{a}{b}$ for A has $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 2$. We do still have to show that no would-be least upper bound $\frac{a}{b}$ for A has $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} > 2$ or $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} < 2$.

Suppose first that $\frac{a}{b} \in \mathbb{Q}$ is an upper bound for A with $2 < \frac{a^2}{b^2}$. We will show that $\frac{a}{b}$ cannot be the least upper bound for A by showing that there is an upper bound $\frac{c}{d} \in \mathbb{Q}$ for A with $2 < \frac{c^2}{d^2} < \frac{a^2}{b^2}$. (Note that by the definition of A, $2 < \frac{c^2}{d^2}$ is enough to ensure that $\frac{c}{d}$ is an upper bound for

A.) We will find such a rational $\frac{c}{d}$ by modifying $\frac{a}{b}$ to make it just a little bit smaller. Observe that if $k > 1$ then $\frac{c}{d} = \frac{ka-1}{kb} = \frac{a-\frac{1}{k}}{b} < \frac{a}{b}$; the trick will be to find an integer $k > 1$ that also ensures that $\frac{c^2}{d^2} = \frac{(ka-1)^2}{(kb)^2} > 2$, *i.e.* such that $(ka-1)^2 > 2(kb)^2$. Now

$$\begin{aligned}(ka-1)^2 > 2(kb)^2 &\iff (ka-1)^2 - 2(kb)^2 > 0\\ &\iff k^2a^2 - 2ka + 1 - 2k^2b^2 > 0\\ &\iff \left(a^2 - 2b^2\right)k^2 - 2ak + 1 > 0,\end{aligned}$$

so the problem comes down whether we can find $k > 1$ such that the polynomial (in k) expression $\left(a^2 - 2b^2\right)k^2 - 2ak + 1$ is greater than zero. Since $2 < \frac{a^2}{b^2}$, we know that $a^2 > 2b^2$, so the coefficient $a^2 - 2b^2$ of k^2 in this polynomial is positive. Since this is the top power of the polynomial, it follows that the polynomial must be positive for large enough k. (Why? :-) Thus, for some large enough integer $k > 0$, if we set $c = ka - 1$ and $d = kb$, we get that $2 < \frac{c^2}{d^2} < \frac{a^2}{b^2}$, so $\frac{c}{d}$ is an upper bound for A that is smaller than $\frac{a}{b}$, as required.

Suppose next that $\frac{a}{b} \in \mathbb{Q}$ is an upper bound for A with $\frac{a^2}{b^2} < 2$, which would mean, by the definition of A, that $\frac{a}{b}$ is the largest element of A. We can show that this impossible by finding $\frac{c}{d} \in A$, *i.e.* with $\frac{c^2}{d^2} < 2$, such that $\frac{a}{b} < \frac{c}{d}$. This can be done with methods very similar to those used in the previous case and are left to the interested reader.

Putting all of this together, the set A has an upper bound, but no least upper bound, in the rational numbers. (Whew!) ■