Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Solutions to Assignment #8
What comes before!

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

Suppose $\mathbb{Q}$ and the usual linear order on the rationals (usually denoted by $<$, or by $<_{\mathbb{Q}}$ when you have other linear orders to keep track of) are defined as they were in class:

- Define the equivalence relation $\approx$ on pairs of integers by
  $(a,b) \approx (c,d)$ if and only if $ad = bc$.
- If (a,b) is a pair of integers with $b \neq 0$, let
  $[(a,b)]_{\approx} = \{\, (c,d) \in \mathbb{Z} \times \mathbb{Z} \mid (a,b) \approx (c,d) \,\}$
  Intuitively, this equivalence class represents the fraction $\frac{a}{b}$.
- Let $\mathbb{Q} = \left\{\, [(a,b)]_{\approx} \mid a,\, b \in \mathbb{Z} \text{ and } b \neq 0 \,\right\}$.
- We can assume that the second coordinate in the pair defining an equivalence class is positive (intuitively, we can assume that the denominator of any fraction is positive). If we stick to such pairs as representatives of equivalence classes, then $[(a,b)]_{\approx} <_{\mathbb{Q}} [(c,d)]_{\approx}$ if and only if $ad <_{\mathbb{Z}} bc$.

In answering the questions below, you may assume that all the familiar properties of the integers, as well as the operations and linear order on the integers, are true.

1. Using these definitions of $\mathbb{Q}$ and $<_{\mathbb{Q}}$, show that $\mathbb{Q}$ has no endpoints, *i.e.* $\mathbb{Q}$ has no smallest and no largest element. *[5]*

SOLUTION. To show that $\mathbb{Q}$ has no left endpoint it suffices to show that for every $q \in \mathbb{Q}$ there is an $r \in \mathbb{Q}$ with $r <_{\mathbb{Q}} q$. Suppose that $q = [(a,b)]_{\approx}$, where we may also suppose that $0 <_{\mathbb{Z}} b$. Let $r = q - 1 = q + (-1) = [(a,b)]_{\approx} + [(-1,1)]_{\approx} = [(a \cdot 1 + b \cdot (-1), b \cdot 1)]_{\approx} = [(a-b,b)]_{\approx}$. (Note that the $a - b$ in the last step is taking place in $\mathbb{Z}$.) Since $0 <_{\mathbb{Z}} b$, we have

$$r = [(a-b,b)]_{\approx} <_{\mathbb{Q}} [(a,b)]_{\approx} = q$$

because

$$(a-b) \cdot b = ab - b^2 <_{\mathbb{Z}} ab = ba\,.$$

Mutatis mutandis†, a very similar argument – which is left to the interested reader! – shows that $\mathbb{Q}$ has no right endpoint either. ■

2. Show that $\mathbb{Q}$ is *countable*, that is, that there is a 1–1 onto function $f : \mathbb{N} \to \mathbb{Q}$. *[5]*

NOTE. Any such function will not play well with the respective arithmetic operations or relations in each number system.

SOLUTION. We gave saw a somewhat informal, but valid, proof in class (2022-11-08). It would be perfectly acceptable to reproduce that argument if one wrote out that lecture's oral explanation of what is going on in the process, or even just to reference the lecture in question. However, such repetition is a little boring, so we'll give another proof, albeit one a little less direct. First, a few useful ways to check if a set A is countable that are usually easier to use than defining a 1–1 onto function $f : \mathbb{N} \to A$ explicitly.

LEMMA 1. Suppose A is an infinite set and $g : A \to \mathbb{N}$ is a 1–1 function. Then A is countable.

PROOF. Let $T = \{ n \in \mathbb{N} \mid n = g(a) \text{ for some } a \in A \}$ be the range of T and let $t_0, t_1, t_2, \ldots$ list T in order of size, *i.e.* t_k denotes the kth element of T. Note that since A is infinite and g is 1–1, T must be infinite as well. Also, since g is 1–1, g has an inverse on T: if $t_k \in T$, then $g^{-1}(t_k) = a$ for the unique $a \in A$ such that $g(a) = t_k$. Note that $g^{-1} : T \to A$ is both 1–1 (Why?) and onto (Also why?).

Define the function $f : \mathbb{N} \to A$ by $f(k) = g^{-1}(t_k)$. Then f is 1–1 and onto since it is a composition of two 1–1 onto functions, namely the enumeration of T, *i.e.* $k \mapsto t_k$, and g^{-1}. Thus A is countable by the definition of countable. □

COROLLARY. An infinite subset of a countable set is also countable.

PROOF. Suppose B is an infinite subset of a countable set A. Then the function $\beta : B \to A$ defined by $\beta(b) = b$ for each $b \in B$ is 1–1, and we know, by the definition of countable, that there

† This translates roughly as "necessary changes having been made".

is a 1–1 onto function $f : \mathbb{N} \to A$. Then the function $f^{-1} \circ \beta : B \to \mathbb{N}$ (recall that the composition is given by $\left(f^{-1} \circ \beta\right)(b) = f^{-1}\left(\beta(b)\right)$ for all $b \in B$) is 1–1, since it is the composition of two 1–1 functions. It follows by Lemma 1 that B is countable. $\square$

Lemma 2. Suppose A is an infinite set and $h : \mathbb{N} \to A$ is an onto function. Then A is countable.

Proof. Define $\varphi : A \to \mathbb{N}$ by, for each $a \in A$, making $\varphi(a)$ be the least $n \in \mathbb{N}$ such that $h(n) = a$. It's not hard to see that φ must be 1–1. By Lemma 1, it follows that A is countable. $\square$

Lemma 3. Suppose $\alpha : A \to B$ is 1–1 and onto. Then if one of A or B is countable, so is the other.

Proof. First, suppose that A is countable, so there is a 1–1 onto function $f : \mathbb{N} \to A$. In this case, the composition $\alpha \circ f$ is a 1–1 onto function $\mathbb{N} \to B$, so B is countable.

Second, suppose B is countable, so there is a 1–1 onto function $g : \mathbb{N} \to B$. In this case, the composition $\alpha^{-1} \circ g$ is a 1–1 onto function $\mathbb{N} \to A$, so A is countable. $\square$

Using these lemmas we will show that $\mathbb{Q}$ is countable in stages:

Zeroth, we know from class that $\mathbb{Z}$ is countable by definition because the function $h : \mathbb{N} \to \mathbb{Z}$ given by $h(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n+1}{2} & n \text{ is odd} \end{cases}$ is 1–1 and onto.

First, the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $g(a,b) = 2^a 3^b$ is 1–1, so $\mathbb{N} \times \mathbb{N}$ is countable by Lemma 1.

Second, $\mathbb{Z} \times \mathbb{Z}$ is countable by Lemma 3 because $\mathbb{N} \times \mathbb{N}$ is countable and the function $H : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ given by $H(n,m) = (h(n), h(m))$ is 1–1 and onto.

Third, $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{\, (a,b) \mid a,\, b \in \mathbb{Z} \text{ and } b \neq 0 \,\}$ is countable by the Corollary to Lemma 1 because it is is an infinite subset of the countable set $\mathbb{Z} \times \mathbb{Z}$.

Fourth, by the definition of $\mathbb{Q}$, the function $e : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ given by $e(a,b) = [(a,b)]_{\approx}$ is onto. Since $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable, there is a 1–1 onto function $f : \mathbb{N} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $e \circ f$ is a function $\mathbb{N} \to \mathbb{Q}$ which is onto because it is the composition of two onto functions. Since $\mathbb{Q}$ is infinite, it follows by Lemma 2 that $\mathbb{Q}$ is countable, as desired. $\blacksquare$