Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Solutions to Assignment #7
What comes before!

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

Suppose $\mathbb{Z}$ has been successfully defined somehow or other, along with the linear order $<$ on $\mathbb{Z}$. However, we have not been given any operations on it except the successor function $S : \mathbb{Z} \to \mathbb{Z}$, which for each $a \in \mathbb{Z}$ gives us the next larger integer, the predecessor function $P : \mathbb{Z} \to \mathbb{Z}$, which for each $a \in \mathbb{Z}$ gives us the next smaller integer, and the negative function $N : \mathbb{Z} \to \mathbb{Z}$, which for each $a \in \mathbb{Z}$ gives us the integer $N(a)$ exactly as far from 0 as a, but on the other side of 0. Informally, of course, $S(a) = a + 1$, $P(a) = a - 1$, and $N(a) = -a$; this has to be informal because we don't yet have the operations of addition and subtraction.

1. Use the given functions to define $+$ and $-$ on $\mathbb{Z}$ inductively. *[7]*

SOLUTION. Before giving the definitions of $+$ and $-$ on $\mathbb{Z}$, it's probably worth sorting out a couple of basic properties of the successor, predecessor, and negative functions, in particular in how they interact.

First, its pretty obvious from the definitions that, for all $a \in \mathbb{Z}$,

$$P\left(S(a)\right) = a = S\left(P(a)\right) \,,$$

that is, P and S are each others inverses. Intuitively, what's going on is that $(a + 1) - 1 = a = (a - 1) + 1$.

Second, for all $a \in \mathbb{Z}$,

$$P\left(N(a)\right) = N\left(S(a)\right) \quad \text{and} \quad S\left(N(a)\right) = N\left(P(a)\right) \,.$$

For example, suppose $a > 0$. Then $S(a)$ is one step from a away from 0, so $N\left(S(a)\right)$ is one step away from $N(a)$ away from 0, which means that $P\left(N(a)\right) = N\left(S(a)\right)$ in this case. Similar arguments take care of the other equation for $a > 0$, and both equations when $a < 0$. The arguments when $a = 0$ are also pretty easy. Intuitively, these equations amount to $-a - 1 = -(a + 1)$ and $-a + 1 = -(a - 1)$.

We can now define $+$ on $\mathbb{Z}$ inductively:

- For all $a \in \mathbb{Z}$, $a + 0 = a$.
- For all $a \in \mathbb{Z}$, if $a + b$ and $a + N(b)$ have been defined for some $b \geq 0$, let $a + S(b) = S(a + b)$ and $a + P(N(b)) = P(a + N(b))$.

Note that this is basically just like how we defined $+$ on $\mathbb{N}$ using the successor function. The main difference is the additional use of the predecessor and negative functions to simultaneously define how addition of negative integers works.

We can define $-$ on $\mathbb{Z}$ inductively in a similar way:

- For all $a \in \mathbb{Z}$, $a - 0 = a$.
- For all $a \in \mathbb{Z}$, if $a - b$ and $a - N(b)$ have been defined for some $b \geq 0$, let $a - S(b) = P(a - b)$ and $a - P(N(b)) = S(a - N(b))$.

Alternatively, we can use the fact that we have already defined addition and have the negative function to define subtraction:

- For all a, $b \in \mathbb{Z}$, let $a - b = a + N(b)$.

We will let the interested reader prove that these definitions actually work and define addition and subtraction properly, since that wasn't asked for ... :-) ■

2. Use your definitions of $+$ and $-$ to show that for all integers $a \in \mathbb{Z}$, $a + N(a) = 0$. *[3]*

Solution. This is a little harder than it looks in one important respect, in that the straightforward ways to prove this fact rely in knowing properties of the operations you want to use, which probably ought to be proven first. In what follows we will assume that addition, in particular, is commutative, *i.e.* $a + b = b + a$ for all a, $b \in \mathbb{Z}$.

We will use induction, in both directions simultaneously, to show that for all $a \in \mathbb{Z}$, $a + N(a) = 0$.

Base Step. ($a = 0$) $0 + N(0) = 0 + 0 = 0$ by the definition of $+$ in **1** above and the fact that $N(0) = 0$. (Since 0 is just as far from 0 as 0 is on the other side of 0 ... :-).

Inductive Hypothesis. ($-k \leq a \leq k$) For some $k \geq 0$, $a + N(a) = 0$ for all a with $N(k) \leq a \leq k$.

Inductive Step. $(a = \pm(k+1))$

First, suppose $a = S(k)$. [This is the induction step in the positive direction.] We will use the properties of the successor, predecessor, and negative functions, the definition of addition given above and the commutativity of addition, and – at the very end – the induction hypothesis:

$$\begin{aligned} a + N(a) &= S(k) + N\left(S(k)\right) = S(k) + P\left(N(k)\right) = P\left(S(k) + N(k)\right) \\ &= P\left(N(k) + S(k)\right) = P\left(S\left(N(k) + k\right)\right) = N(k) + k \\ &= k + N(k) = 0 \end{aligned}$$

Second, suppose $a = P\left(N(k)\right)$. [This is the induction step in the negative direction.] Again, we will use the properties of the successor, predecessor, and negative functions, the definition of addition given above and the commutativity of addition, and – at the very end – the induction hypothesis:

$$\begin{aligned} a + N(a) &= P\left(N(k)\right) + N\left(P\left(N(k)\right)\right) = P\left(N(k)\right) + N\left(N\left(S(k)\right)\right) \\ &= P\left(N(k)\right) + S(k) = S\left(P\left(N(k)\right) + k\right) = S\left(k + P\left(N(k)\right)\right) \\ &= S\left(P\left(k + N(k)\right)\right) = k + N(k) = 0 \end{aligned}$$

It follows by induction that $a + N(a) = 0$ for all $a \in \mathbb{Z}$. ■