Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Solutions to Assignment #4
The Fibonacci Sequence and the Golden Ratio

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

Leonardo of Pisa (*c.* 1170 A.D. – *c.* 1250 A.D.), nowadays commonly known as Fibonacci, is usually considered to be the best European mathematician of the Middle Ages. He is most famous nowadays for a problem in his book *Liber Abaci* (1202 A.D., revised *c.* 1227 A.D.) that involved what is now called the Fibonacci sequence.[†] It is usually defined recursively as follows:

$$f_0 = 0,\ f_1 = 1, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

This sequence turns up in surprising places, such as the arrangement of seeds in a sunflower. It also has connections with the so-called *Golden Ratio*, which can be defined as the positive real number φ satisfying the equation $\varphi = \dfrac{\varphi+1}{\varphi} = 1+\dfrac{1}{\varphi}$, which also turns up in surprising places. This assignment will have you work through a couple of bits of this connection.

1. Show that $\varphi = \dfrac{1+\sqrt{5}}{2}$. *[1]*

SOLUTION. Multiplying by φ on both sides of $\varphi = \dfrac{\varphi+1}{\varphi}$ gives us the quadratic equation $\varphi^2 = \varphi + 1$. Rearranging this to $\varphi^2 - \varphi - 1 = 0$ lets us apply the quadratic formula:

$$\varphi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since $\dfrac{1+\sqrt{5}}{2} > 0$ and $\dfrac{1-\sqrt{5}}{2} < 0$, and we are told that φ is "the positive real number φ satisfying the equation $\varphi = \dfrac{\varphi+1}{\varphi}$", it follows that $\varphi = \dfrac{1+\sqrt{5}}{2}$. ■

2. Use induction to show that $f_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}}$ for all $n \geq 0$. *[6]*

SOLUTION THE FIRST. Let's put the problem in as concrete terms as we can. Observe that

$$-\frac{1}{\varphi} = -\frac{1}{\frac{1+\sqrt{5}}{2}} = -\frac{2}{1+\sqrt{5}} = -\frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{(-2)\left(1-\sqrt{5}\right)}{1-5} = \frac{1-\sqrt{5}}{2},$$

[†] The sequence was already known to Indian mathematicians by *c.* 700 A.D., and possibly several centuries earlier, depending on how one interprets some ambiguous language in the relevant texts.

which happens to be the negative root of the equation of which the golden ratio is the positive root. This means that we are being asked to show that for all $n \geq 0$,

$$
\begin{aligned}
f_n &= \frac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}} = \frac{\varphi^n - (-\varphi)^{-n}}{\varphi + \frac{1}{\varphi}} = \frac{\varphi^n - (-\varphi)^{-n}}{\varphi - \frac{-1}{\varphi}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],
\end{aligned}
$$

which we shall do by induction on n. Note that we require two base steps, one for each of $f_0 = 0$ and $f_1 = 1$. Here goes:

Base Step ($n = 0$): $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right] = \frac{1}{\sqrt{5}}[1-1] = \frac{0}{\sqrt{5}} = 0 = f_0$

Base Step ($n = 1$): $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} = 1 = f_1$

Inductive Hypothesis ($n \leq k$): $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ for all $n \geq 0$ with $n \leq k$ (with $k \geq 1$).

Inductive Step ($n = k+1$): We need to show that $f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$.

Before we start, observe that $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$ and that $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$. Here we go:

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \qquad \text{Here we apply the Inductive Hypothesis:} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2} + 1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} + 1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} \quad \text{... and here our observation:} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^2\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}
\end{aligned}
$$

It thus follows by induction that, in less concrete terms, $f_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}}$ for all $n \geq 0$. ■

SOLUTION THE SECOND.[‡] As in the solution above, we note that $\psi = -\varphi^{-1} = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}$ is, along with $\varphi = \frac{1+\sqrt{5}}{2}$, also a solution of the quadratic equation $x^2 = x + 1$. It follows that for any integer m we have

$$\varphi^{m+2} = \varphi^m \varphi^2 = \varphi^m(\varphi + 1) = \varphi^{m+1} + \varphi^m \quad \text{and}$$
$$(-\varphi)^{-(m+2)} = \psi^{m+2} = \psi^m \psi^2 = \psi^m(\psi + 1) = \psi^{m+1} + \psi^m = (-\varphi)^{-(m+1)} + (-\varphi)^{-m}\,.$$

This will give us a much shorter argunent at the inductive step when we show that $f_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}}$ for all $n \geq 0$ by induction on n:

Base Step ($n = 0$): $\frac{\varphi^0 - (-\varphi)^{-0}}{\varphi + \varphi^{-1}} = \frac{1-1}{\varphi + \varphi^{-1}} = 0 = f_0$

Base Step ($n = 1$): $\frac{\varphi^1 - (-\varphi)^{-1}}{\varphi + \varphi^{-1}} = \frac{\varphi + \varphi^{-1}}{\varphi + \varphi^{-1}} = 1 = f_1$

Inductive Hypothesis ($n \leq k$): $f_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}}$ for all $n \geq 0$ with $n \leq k$ (with $k \geq 1$).

Inductive Step ($n = k + 1$):

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \qquad \text{Here we apply the Inductive Hypothesis:} \\
&= \frac{\varphi^k - (-\varphi)^{-k}}{\varphi + \varphi^{-1}} + \frac{\varphi^{k-1} - (-\varphi)^{-(k-1)}}{\varphi + \varphi^{-1}} \\
&= \frac{\left(\varphi^k + \varphi^{k-1}\right) - \left((-\varphi)^{-k} + (-\varphi)^{-(k-1)}\right)}{\varphi + \varphi^{-1}} \quad \ldots \text{ and here the observation:} \\
&= \frac{\varphi^{k+1} - (-\varphi)^{-(k+1)}}{\varphi + \varphi^{-1}}
\end{aligned}$$

It thus follows by induction that $f_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}}$ for all $n \geq 0$. ■

3. Show that $\displaystyle\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \varphi$. *[3]*

SOLUTION. One *could* do this with epsilonics, but we'll rely on our knowledge of limits from first-year calculus and question **2** above. One quick preliminary: appealing to a calculator tells us that $\left|(-\varphi)^{-1}\right| = \frac{1}{\varphi} < 1$ because $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6$, from which it follows

[‡] The essential parts of this clever solution are due to a student in the class, Thelonious Blake, who ran his solution by me before handing it in.

that $\lim_{n\to\infty}(-\varphi)^{-n} = \lim_{n\to\infty}\left(-\varphi^{-1}\right)^n = 0$ and that $\lim_{n\to\infty}\varphi^{-n} = \lim_{n\to\infty}\left(\varphi^{-1}\right)^n = 0$.

$$\begin{aligned}
\lim_{n\to\infty}\frac{f_{n+1}}{f_n} &= \lim_{n\to\infty}\frac{\frac{\varphi^{n+1}-(-\varphi)^{-(n+1)}}{\varphi+\varphi^{-1}}}{\frac{\varphi^{n}-(-\varphi)^{-n}}{\varphi+\varphi^{-1}}} = \lim_{n\to\infty}\frac{\varphi^{n+1}-(-\varphi)^{-(n+1)}}{\varphi^{n}-(-\varphi)^{-n}} \\
&= \lim_{n\to\infty}\varphi\cdot\frac{\varphi^{n}-(-1)^{n+1}\varphi^{-(n+2)}}{\varphi^{n}-(-\varphi)^{-n}} \\
&= \varphi\cdot\lim_{n\to\infty}\frac{\varphi^{n}-(-1)^{n+1}\varphi^{-(n+2)}}{\varphi^{n}-(-\varphi)^{-n}} = \varphi\cdot 1 = \varphi\,,
\end{aligned}$$

since $\varphi^{-n}\to 0$ and $(-\varphi)^{-n}\to 0$ as $n\to\infty$. ■