Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Assignment #3
Sherlock Holmes in (Heaven ∨ Hell)?

1. Suppose A and B are atomic formulas of propositional logic. For each of the following formulas,

a. $(A \wedge B)$ *[2]* and

b. $(A \leftrightarrow B)$ *[2]*,

find a formula truth-table equivalent to it using only the symbols A, B, $\neg$, $\rightarrow$, (, and), and verify that your formula does the job. You may use each symbol as many times as you like in each formula.

SOLUTIONS. **a.** One formula using only the symbols A, B, $\neg$, $\rightarrow$, (, and) that is truth-table equivalent to $(A \wedge B)$, among other possibilities, is $(\neg(A \rightarrow (\neg B)))$. Here is a truth table verifying this:

A	B	$(\neg B)$	$(A \rightarrow (\neg B))$	$(\neg(A \rightarrow (\neg B)))$	$(A \wedge B)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Observe that the last two columns are identical except for the formulas at the top ... □

b. Intuitively, $(A \leftrightarrow B)$ means $(A \rightarrow B) \wedge (B \rightarrow A))$, so the most straightforward approach is to leverage part **a** to write the second formula entirely in terms of the symbols A, B, $\neg$, $\rightarrow$, (, and). This gives us the pretty ugly formula $(\neg((A \rightarrow B) \rightarrow (\neg(B \rightarrow A))))$, but it does work:

A	B	$(A \rightarrow B)$	$(B \rightarrow A)$	$(\neg(B \rightarrow A))$	$((A \rightarrow B) \rightarrow (\neg(B \rightarrow A)))$	$(\neg((A \rightarrow B) \rightarrow (\neg(B \rightarrow A))))$	$(A \leftrightarrow B)$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	F	F	T	T

Whew! Again, observe that the last two columns are identical except for the formulas at the top ... ■

2. Construct deductions in (our version of) propositional logic from the given hypotheses to the given conclusions in each of the following cases:

a. Hypotheses: $(\alpha \to \beta)$ and $(\beta \to \gamma)$; Conclusion: $\cancel{(\alpha \to \beta)}$ $(\alpha \to \gamma)$. *[3]*

b. Hypotheses: none; Conclusion: $((\neg(\neg\varphi)) \to \varphi)$. *[3]*

NOTE. For your convenience for question **2**, a summary of our official system of propositional logic is on page ~~2~~ 4.

SOLUTIONS. **a.** Besides the hypotheses (or premisses) $(\alpha \to \beta)$ and $(\beta \to \gamma)$, we will only need axioms **A1** and **A2** in our deduction below.

1.	$((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$	A2
2.	$((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma)))$	A1
3.	$(\beta \to \gamma)$	Hypothesis
4.	$(\alpha \to (\beta \to \gamma))$	2, 3 MP
5.	$((\alpha \to \beta) \to (\alpha \to \gamma))$	1, 4 MP
6.	$(\alpha \to \beta)$	Hypothesis
7.	$(\alpha \to \gamma)$	5, 6 MP

Thus $\{ (\alpha \to \beta), (\beta \to \gamma) \} \vdash (\alpha \to \beta)$, as desired. □

b. We will divide and conquer by using the deductions we have already done in class of $\vdash (\alpha \to \alpha$ (see page 4 below, let's call it Lemma **I**) and in part **a**, plus the lemma below, as what amount to additional rules of procedure to reduce the length of the deduction we need to produce to verify that $\vdash ((\neg(\neg\varphi)) \to \varphi)$.

LEMMA **II**. For any formulas α, β, and γ,

$$\{ (\alpha \to (\beta \to \gamma)) , \beta \} \vdash (\alpha \to \gamma).$$

Proof. Here is the deduction:

1.	$((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$	A2
2.	$(\alpha \to (\beta \to \gamma))$	Hypothesis
3.	$((\alpha \to \beta) \to (\alpha \to \gamma))$	1, 2 MP
4.	$(\beta \to (\alpha \to \beta))$	A1
5.	β	Hypothesis
6.	$(\alpha \to \beta)$	4, 5 MP
7.	$(\alpha \to \gamma)$ 3, 6 MP	

Thus $\{ (\alpha \to (\beta \to \gamma)) , \beta \} \vdash (\alpha \to \gamma)$, as desired. □

We will use part **a** and Lemmas **I** and **II** to keep the deduction to verify that $\vdash ((\neg(\neg\varphi)) \to \varphi)$, for any formula φ, tolerably short and comprehensible:

1.	$(((\neg\varphi) \to (\neg(\neg\varphi))) \to (((\neg\varphi) \to (\neg\varphi)) \to \varphi))$	A3
2.	$((\neg(\neg\varphi)) \to ((\neg\varphi) \to (\neg(\neg\varphi))))$	A1
3.	$((\neg(\neg\varphi)) \to (((\neg\varphi) \to (\neg\varphi)) \to \varphi))$	1, 2 Part **a**
4.	$((\neg\varphi) \to (\neg\varphi))$	Lemma **I**
5.	$((\neg(\neg\varphi)) \to \varphi)$	3, 4 Lemma **II**

Since Lemma **I** tells us that for any formula α there is a deduction of $(\alpha \to \alpha)$ using just the logical axioms and Modus Ponens, *i.e.* that $\vdash (\alpha \to \alpha)$, we can appeal to it to justify line 4 in the deduction above. Technically, this is a form of shorthand; if we were truly formal we would insert the deduction in Lemma **I**, with α replaced by $(\neg\varphi)$ throughout, in place of line 4, renumbering the lines of the overall deduction accordingly.

Similarly, if were that formal, we would replace the current line 3 with the deduction from part **a**, with α, β, and γ replaced by $(\neg(\neg\varphi))$, $(((\neg\varphi) \to (\neg\varphi)) \to \varphi)$, and $(((\neg\varphi) \to (\neg\varphi)) \to \varphi)$, respectively, and renumber the overall deduction accordingly. Note that lines 1 and 2 of the current deduction then correspond to the hypotheses required in part **a**.

Again, if were that formal, we would replace the current line 5 with the deduction in Lemma **II**, with the formulas α, β, and γ replaced by $(\neg(\neg\varphi))$, $((\neg\varphi) \to (\neg\varphi))$, and φ, respectively. Note that lines 3 and 4 of the current deduction then correspond to the hypotheses required in the deduction of Lemma **II**. $\blacksquare$

Our official system of propositional logic, as we developed it in class:

- The symbols of the language are:
  1. The atomic formulas: A_1, A_2, A_3, ...
  2. The connectives: $\neg$ and $\rightarrow$.[†]
  3. The grouping symbols: (and).
- The formulas of the language are defined as follows:
  1. Every atomic formula is a formula.
  2. If α is a formula, so is $(\neg\alpha)$.
  3. If α and β are formulas, so is $(\alpha \rightarrow \beta)$.
  4. Nothing else is a formula.
- The logical axioms are defined as follows. Suppose α, β, and γ are any formulas of the language. Then the following are logical axioms:

  A1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$

  A2. $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$

  A3. $(((\neg\beta) \rightarrow (\neg\alpha)) \rightarrow (((\neg\beta) \rightarrow \alpha) \rightarrow \beta))$
- The sole rule of prodecure is Modus Ponens (MP): Given formulas of the language α and $(\alpha \rightarrow \beta)$, we may infer β.
- If Γ is a set, possibly empty, of formulas of the language and α is a formula of the language, then a deduction or proof of α from the set of hypotheses Γ, written as $\Gamma \vdash \alpha$ (or as $\vdash \alpha$ if $\Gamma = \emptyset$), is a finite sequence of formulas of the language, say $\varphi_1, \varphi_2, \ldots, \varphi_n$, with φ_n being α, such that each φ_k is
  1. a hypothesis, *i.e.* $\varphi_k \in \Gamma$, or
  2. a logical axiom, or
  3. follows from preceding formulas in the sequence by Modus Ponens, *i.e.* there are $i, j < k$ such that φ_j is $(\varphi_i \rightarrow \varphi_k)$.

The example of a formal deduction that we did in class [the Lemma **I** referred to in the solution to **2a** above, was of $\vdash (\alpha \rightarrow \alpha)$, where α could be any formula of the language :

1.	$((\alpha \rightarrow ((\alpha \rightarrow \alpha) \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\alpha \rightarrow \alpha)) \rightarrow (\alpha \rightarrow \alpha)))$	A2
2.	$(\alpha \rightarrow ((\alpha \rightarrow \alpha) \rightarrow \alpha))$	A1
3.	$((\alpha \rightarrow (\alpha \rightarrow \alpha)) \rightarrow (\alpha \rightarrow \alpha))$	1, 2 MP
4.	$(\alpha \rightarrow (\alpha \rightarrow \alpha))$	A1
5.	$(\alpha \rightarrow \alpha)$	3, 4 MP

† The often-used connectives $\vee$, $\wedge$, and $\leftrightarrow$ are abbreviations for constructions using only the official connectives, as in question **1**.