Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Solutions to Assignment #2
Order out of chaos?

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

We discussed some basic theory, notation, and definitions in class. In this assignment you will try to define the basic concept of "ordered pair" using only unordered sets. Here is the problem:

> *Given arbitrary sets a and b, define the ordered pair (a, b) using only the symbols $\{$ and $\}$ and , (i.e. the comma), as well as, of course, a and b.*

Whatever your definition is, it is essential that it allow one to distinguish what is in the first coordinate and what is in the second, no matter what the sets a and b actually are. So, for example, $\{a, b\}$ does not work as a definition of (a, b) because sets are inherently unordered: since $\{a, b\}$ and $\{b, a\}$ have the same elements, they are the same set. In other words, you can't tell which element is supposed to come first.

1. Solve the problem. *[5]*

SOLUTION. The most commonly used definition of (a, b) is $\{\,\{\,a\,\},\,\{\,a,\,b\,\}\,\}$, due to Kazimierz Kuratowski (1896-1980). Of course, others are possible. ■

2. Explain why your solution actually works, as precisely and completely as you can. *[5]*

SOLUTION. We need to show that if $(a, b) = (c, d)$, then $a = c$ and $b = d$. (This simultaneously shows that we can pick out the first *vs.* second coordinates reliably and that the ordered pair is uniquely specified by specifying the first and second coordinates.)

Suppose that $\{\,\{\,a\,\},\,\{\,a,\,b\,\}\,\} = (a, b) = (c, d) = \{\,\{\,c\,\},\,\{\,c,\,d\,\}\,\}$. It follows that $\{\,a\,\} \in \{\,\{\,c\,\},\,\{\,c,\,d\,\}\,\}$, so either $\{\,a\,\} = \{\,c\,\}$ or $\{\,a\,\} = \{\,c,\,d\,\}$.

In the first case, if $\{\,a\,\} = \{\,c\,\}$, then we must have $a = c$, and hence that $\{\,a,\,b\,\} = \{\,c,\,d\,\} = \{\,a,\,d\,\}$, from which it now follows that $b = d$ as well.

In the second case, if $\{ a \} = \{ c, d \}$, we must have $a = c = d$. $\{ a, b \}$ must then be equal to $\{ c \}$, or equal to $\{ a, b \} = \{ c, d \} = \{ c, c \} = \{ c \}$, and it follows that $a = b = c$ either way. Thus $a = b = c = d$ if $\{ a \} = \{ c, d \}$.

In either case, we have $a = c$ and $b = d$, as required. ■

Note. For question **2**, please keep the four Cs of writing proofs (and mathematics in general) in mind. A proof (or argument, or definition) should ideally be, in order of priority:

1. Correct
2. Complete
3. Clear
4. Concise