Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022
Solutions to Assignment #10
Getting real!

Please give your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

Recall from class that a set S of rational numbers is a *schnitt* if

(1) $S \neq \emptyset$ and $S \neq \mathbb{Q}$,
(2) S is *downward closed*, *i.e.* whenever $s \in S$, $q \in \mathbb{Q}$, and $q <_{\mathbb{Q}} s$, it follows that $q \in S$,
and (3) S has no largest element, *i.e.* for every $s \in S$, there is a $t \in S$ such that $s <_{\mathbb{Q}} t$.

We then defined the real numbers to be the schnitts, *i.e.* officially $\mathbb{R} = \{\, S \subset \mathbb{Q} \mid S \text{ is a schnitt} \,\}$, and proceeded to define addition of real numbers, *a.k.a.* schnitts, by $S +_{\mathbb{R}} T = \{\, s + t \mid s \in S \text{ and } t \in T \,\}$.

1. Check that $0_{\mathbb{R}} = \{\, q \in \mathbb{Q} \mid q <_{\mathbb{Q}} 0_{\mathbb{Q}} \,\}$ is indeed a schnitt. *[3]*

SOLUTION. We need to check that $0_{\mathbb{R}}$ satisfies the defining conditions for being a schnitt.

i. First, $0_{\mathbb{R}} \neq \emptyset$ since, by its definition, $-1_{\mathbb{Q}} = \frac{-1}{1} = [(-1,1)]_\sim \in 0_{\mathbb{R}}$ because $-1_{\mathbb{Q}} <_{\mathbb{Q}} 0_{\mathbb{Q}}$. Second, $0_{\mathbb{R}} \neq \mathbb{Q}$ since, by its definition, $1_{\mathbb{Q}} = \frac{1}{1} = [(1,1)]_\sim \notin 0_{\mathbb{R}}$ because $1_{\mathbb{Q}} \not< 0_{\mathbb{Q}}$.

ii. $0_{\mathbb{R}}$ is downward closed: Suppose $q \in 0_{\mathbb{R}}$ and $r \in \mathbb{Q}$ with $r <_{\mathbb{Q}} q$. Since, by definition, $q \in 0_{\mathbb{R}}$ means that $q <_{\mathbb{Q}} 0_{\mathbb{Q}}$, and $<_{\mathbb{Q}}$ is transitive, it follows that $r <_{\mathbb{Q}} 0_{\mathbb{Q}}$, so $r \in 0_{\mathbb{R}}$ by definition, as required.

iii. $0_{\mathbb{R}}$ has no largest element: Suppose $q = [(a,b)]_\sim \in 0_{\mathbb{R}}$. By definition, this means that $q <_{\mathbb{Q}} 0_{\mathbb{Q}}$, but then $q <_{\mathbb{Q}} q/2 = [(a,2b)]_\sim <_{\mathbb{Q}} 0_{\mathbb{Q}}$, so $q/2 \in 0_{\mathbb{R}}$ by definition and q is less than $q/2$.

Thus $0_{\mathbb{R}}$ is indeed a schnitt. ■

2. Show that if S is any schnitt, then $S +_{\mathbb{R}} 0_{\mathbb{R}} = S$. *[7]*

SOLUTION. By definition,

$$S +_{\mathbb{R}} 0_{\mathbb{R}} = \{\, s + q \mid s \in S \text{ and } q \in 0_{\mathbb{R}} \,\} = \{\, s + q \mid s \in S \text{ and } q <_{\mathbb{Q}} 0_{\mathbb{Q}} \,\},$$

which we know from class is a schnitt since addition of schnitts makes schnitts. We will show that $S +_{\mathbb{R}} 0_{\mathbb{R}} = S$ by showing that $S +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq S$ and $S \subseteq S +_{\mathbb{R}} 0_{\mathbb{R}}$.

First, suppose $s + q \in S +_{\mathbb{R}} 0_{\mathbb{R}}$ for some $s \in S$ and $q <_{\mathbb{Q}} 0_{\mathbb{Q}}$. Then $s + q <_{\mathbb{Q}} q$, so $s + q \in S$ by the downward closure of the schnitt S. Thus $S +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq S$.

Second, suppose $s \in S$. Since S is a schnitt, it has no largest element, so there some $t \in S$ with $s <_{\mathbb{Q}} t$. Then $s - t <_{\mathbb{Q}} 0_{\mathbb{Q}}$, so $s - t \in 0_{\mathbb{R}}$ by definition, and hence $s = t + (s - t) \in S +_{\mathbb{R}} 0_{\mathbb{R}}$ by the definition of $+_{\mathbb{R}}$. Thus $S \subseteq S +_{\mathbb{R}} 0_{\mathbb{R}}$.

Since they have the same elements, it follows that $S +_{\mathbb{R}} 0_{\mathbb{R}} = S$. $\blacksquare$