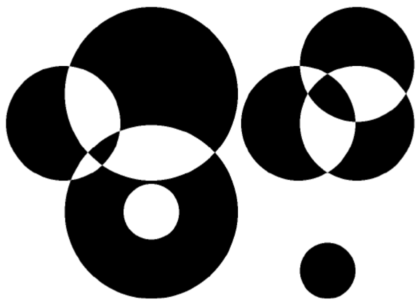
Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Take-Home Final Examination
Due on Friday, 16 December, 2022.*

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. In particular, you may use all the resources this course has on Blackboard and on its archive page. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Proposition. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Suppose finitely many, possibly overlapping, circles are drawn in the plane, dividing it into regions whose borders are made up of circular arcs. Show that the regions can be coloured using white and black so that no two regions sharing a common border have the same colour.



2. Recall that the inhabitants of the Island of Knights and Knaves are either knights, who always tell the truth, or knaves, who always lie. While visiting the Island you meet nine inhabitants: Zippy, Betty, Bill, Homer, Dave, Zed, Bob, Marge and Carl. Zippy says that Bill is a knave. Betty says that Bill could say that Carl is a knight. Bill tells you that Homer and Marge are different. Homer claims, "Either Bob is a knave or Carl is a knave." Dave says that Betty and Zed are knaves. Zed claims that either Bill is a knave or Dave is a knave. Bob says, "Marge could say that I am a knave." Marge claims, "I am a knight or Zippy is a knight." Carl tells you, "Either Zed is a knave or Bob is a knight."

Determine, as best you can, which of the nine are knights and which are knaves.

3. Recall that an integer said to be a square if it is equal to n^2 for some integer n. Show that the digit in the 1s place in the decimal expansion of a square could be 0, 1, 4, 5, 6, or 9, but cannot not be 2, 3, 7, or 8.

4. Recall that $0_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q <_{\mathbb{Q}} 0_{\mathbb{Q}} \}$ is the schnitt representing 0 in $\mathbb{R}$ and suppose S is a schnitt. Show that $(-S) + S = 0_{\mathbb{R}}$.

[There are more exam questions on page 2.]

* You may submit your solutions on paper or via Blackboard, or – as a last resort! – by email to the instructor at `sbilaniuk@trentu.ca`.

[You did do the questions on page 1, or at least you intend to later, right?]

Part Quantifier. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Define the logical connective $\downarrow$ via the following truth table:

A	B	$A \downarrow B$
T	T	F
T	F	F
F	T	F
F	F	T

Show how to write formulas equivalent to $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, and $A \leftrightarrow B$ using just the connective $\downarrow$. (You may use it more than once in each formula, of course.)

6. Suppose $\{a_n\} = \{\, a_n \mid n \geq 0 \,\}$ is a sequence such that $\lim_{n\to\infty} a_n = L$. For each $n \geq 0$, let $b_n = \operatorname{lub}\{\, a_k \mid k \geq n \,\}$. Show that:

a. The definition of b_n makes sense for each $n \geq 0$. *[5]* **b.** $\lim_{n\to\infty} b_n = L$ *[5]*

7. Prove the cancellation law for addition in the natural numbers: for all a, b, $c \in \mathbb{N}$, if $a + c = b + c$, then $a = b$. (You may assume, if you wish, that addition on the natural numbers is associative and commutative, and that 0 behaves like it should.)

8. Suppose $A \neq \emptyset$ is a set and $f : A \to A$ is a 1-1 function with $B = \{\, f(a) \mid a \in A \,\} \neq A$. Show that A cannot be finite, *i.e.* it does not have exactly n elements for some $n \in \mathbb{N}$.

9. Suppose $n \in \mathbb{N}$ and $n \geq 2$. Show that:

a. $41^n - 1$ is not prime. *[1]* **b.** If n is not prime, then neither is $2^n - 1$. *[9]*

10. Show that $\mathbb{N}^3 = \{\, (a, b, c) \mid a,\ b,\ c \in \mathbb{N} \,\}$ is countable.

11. *Pentominoes* are shapes obtained by gluing five 1×1 squares together full edge to full edge. (Think Tetris pieces, except with five squares instead of four.) Two pentominoes that can be made congruent via flips or rotations are considered to be the same. Find all twelve pentominoes and an arrangement of all twelve, used once each, into a 5×12 rectangle.

[Total = 80]

Part Rhyme and Reason. Bonus problems!

α. Write an original poem about logic or mathematics. *[1]*

β. Draw a configuration of circles and a colouring of the regions they create that shows that question **1** on this exam could be a Mickey Mouse problem. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE A GOOD BREAK!