Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2022
Assignment #9 – (Non-)Completeness
Due on Friday, 18 November.[*]

Please give your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

As in Assignment #8, suppose $\mathbb{Q}$ and the usual linear order on the rationals (usually denoted by $<$, or by $<_{\mathbb{Q}}$ when you have other linear orders to keep track of) are defined as they were in class:

- Define the equivalence relation $\approx$ on pairs of integers by
  $(a,b) \approx (c,d)$ if and only if $ad = bc$.
- If (a,b) is a pair of integers with $b \neq 0$, let
  $[(a,b)]_{\approx} = \{\, (c,d) \in \mathbb{Z} \times \mathbb{Z} \mid (a,b) \approx (c,d) \,\}$
  Intuitively, this equivalence class represents the fraction $\frac{a}{b}$.
- Let $\mathbb{Q} = \left\{\, [(a,b)]_{\approx} \mid a,\, b \in \mathbb{Z} \text{ and } b \neq 0 \,\right\}$.
- We can assume that the second coordinate in the pair defining an equivalence class is positive (intuitively, we can assume that the denominator of any fraction is positive). If we stick to such pairs as representatives of equivalence classes, then $[(a,b)]_{\approx} <_{\mathbb{Q}} [(c,d)]_{\approx}$ if and only if $ad <_{\mathbb{Z}} bc$.

To all this we add the following idea: a linear order is *complete* if every non-empty subset that has an upper bound (which need not be in the subset) has a least upper bound (which also need not be in the subset).

In answering the questions below, you may assume that all the familiar properties of the integers, as well as the operations and linear order on the integers, are true.

1. Using these definitions of $\mathbb{Q}$ and $<$, show that $\mathbb{Q}$ is not complete, *i.e.* show that there is a non-empty subset $A \subset \mathbb{Q}$ with an upper bound in $\mathbb{Q}$, but no least upper bound in $\mathbb{Q}$. *[10]*

Note. Some non-empty subsets of $\mathbb{Q}$ do have least upper bounds. For example, $\left\{\, q \in \mathbb{Q} \mid q < \frac{5}{3} \,\right\}$ has least upper bound $\frac{5}{3} = [(5,3)]_{\approx}$.

Hint: Informally, think of the rationals as a part of the real number line.

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[*] You may submit your solutions on paper or via Blackboard, or – as a last resort! – by email to the instructor at `sbilaniuk@trentu.ca`.