Mathematics 2200H – Mathematical Reasoning

TRENT UNIVERSITY, Fall 2022

Assignment #6
The Littlest Field

*Due on Friday, 21 October.**

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

$\mathbb{Z}_2$ is the system of integers modulo 2. Roughly, this means that all odd numbers are equal to 1 and all even numbers are equal to 0 when you do arithmetic. (Think of it as keeping time on a rapidly spinning planet with a two-hour day... :-) A little more technically, $\mathbb{Z}_2 = \{0, 1\}$, with the operations $+$ and $\cdot$ given by the following tables:

$+$	0	1		$\cdot$	0	1
0	0	1	and	0	0	0
1	1	0		1	0	1

1. **a.** For each $a \in \mathbb{Z}_2$, what is $-a$, assuming that such exists? *[0.5]*
    b. For each $a \in \mathbb{Z}_2$, what is a^{-1}, assuming that such exists? *[0.5]*

2. Show that $+$ and $\cdot$ on $\mathbb{Z}_2$ are both associative and commutative, and that the distributive law holds. *[4]*

3. Use your answers to **1** and **2** to verify that $\mathbb{Z}_2$ is a *field*†. *[1]*

4. $\mathbb{Z}_2^2$ is the usual two-dimensional vector space over $\mathbb{Z}_2$, in the same way that $\mathbb{R}^2$ is the usual two-dimensional vector space over the field of real numbers $\mathbb{R}$. What are all the subspaces of $\mathbb{Z}_2^2$? *[4]*

Hint: The most complete way to answer the last question is to simply write out each subspace as a set of vectors.