Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2022

Assignment #3
Sherlock Holmes in (Heaven $\vee$ Hell)?
*Due on Friday, 30 September.**

Please your complete reasoning in your solution. Recall that, unless stated otherwise on a given assignment, you are permitted to work together and look things up, so long as you write up your solution by yourself and acknowledge all sources and help that you ended up using.

1. Suppose A and B are atomic formulas of propositional logic. For each of the following formulas,

a. $(A \wedge B)$ *[2]* and

b. $(A \leftrightarrow B)$ *[2]*,

find a formula truth-table equivalent to it using only the symbols A, B, $\neg$, $\rightarrow$, (, and), and verify that your formula does the job. You may use each symbol as manytimes as you like in each formula.

2. Construct deductions in (our version of) propositional logic from the given hypotheses to the given conclusions in each of the following cases:

a. Hypotheses: $(\alpha \rightarrow \beta)$ and $(\beta \rightarrow \gamma)$; Conclusion: $(\alpha \rightarrow \beta)$. *[3]*

b. Hypotheses: none; Conclusion: $((\neg(\neg\varphi)) \rightarrow \varphi)$. *[3]*

NOTE. For your convenience for question **2**, a summary of our official system of propositional logic is on page 2.

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* You may submit your solutions on paper or via Blackboard, or – as a last resort! – by email to the instruictor at `sbilaniuk@trentu.ca` .

Our official system of propositional logic, as we developed it in class:

- The symbols of the language are:
  1. The atomic formulas: A_1, A_2, A_3, $\dots$
  2. The connectives: $\neg$ and $\to$.[†]
  3. The grouping symbols: (and).
- The formulas of the language are defined as follows:
  1. Every atomic formula is a formula.
  2. If α is a formula, so is $(\neg\alpha)$.
  3. If α and β are formulas, so is $(\alpha \to \beta)$.
  4. Nothing else is a formula.
- The logical axioms are defined as follows. Suppose α, β, and γ are any formulas of the language. Then the following are logical axioms:

  A1. $(\alpha \to (\beta \to \alpha))$

  A2. $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$

  A3. $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$
- The sole rule of prodecure is Modus Ponens (MP): Given formulas of the language α and $(\alpha \to \beta)$, we may infer β.
- If Γ is a set, possibly empty, of formulas of the language and α is a formula of the language, then a deduction or proof of α from the set of hypotheses Γ, written as $\Gamma \vdash \alpha$ (or as $\vdash \alpha$ if $\Gamma = \emptyset$), is a finite sequence of formulas of the language, say φ_1, φ_2, $\dots$, φ_n, with φ_n being α, such that each φ_k is
  1. a hypothesis, *i.e.* $\varphi_k \in \Gamma$, or
  2. a logical axiom, or
  3. follows from preceding formulas in the sequence by Modus Ponens, *i.e.* there are $i, j < k$ such that φ_j is $(\varphi_i \to \varphi_k)$.

The example of a formal deduction that we did in class was of $\vdash (\alpha \to \alpha)$, where α could be any formula of the language:

1.	$((\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)))$	A2
2.	$(\alpha \to ((\alpha \to \alpha) \to \alpha))$	A1
3.	$((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$	1, 2 MP
4.	$(\alpha \to (\alpha \to \alpha))$	A1
5.	$(\alpha \to \alpha)$	3, 4 MP

[†] The often-used connectives $\vee$, $\wedge$, and $\leftrightarrow$ are abbreviations for constructions using only the official connectives, as in question **1**.