Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Assignment #6
Cancellation and Distributive Laws
Due on Friday, 22 October.

Recall from class that addition on the natural numbers was defined by recursion with the help of the successor function S as follows:

- For all $n \in \mathbb{N}$, $n + 0 = n$.
- For all $n \in \mathbb{N}$, if $n + k$ has been defined, then $n + S(k) = S(n + k)$.

Since $\mathbb{N}$ lacks negatives, one can't really define the usual notion of subtraction in a general way, costing us a useful tool for solving equations. The following fact lets us compensate for this, at least to some extent.

CANCELLATION LAW (for $+$ on $\mathbb{N}$): For all n, m, $k \in \mathbb{N}$, if $n + k = m + k$, then $n = m$.

1. Prove the Cancellation Law for $+$ on $\mathbb{N}$. *[5]*

SOLUTION. We will proceed by induction on k.

Base Step. $[k = 0]$ Suppose $n+0 = m+0$. Then, by the definition of addition, $n = n+0 = m + 0 = m$.

Induction Hypothesis. Assume that for some $k \geq 0$ we have, for all n, $m \in \mathbb{N}$, that if $n + k = m + k$, then $n = m$.

Induction Step. $[k \to S(k)]$ Suppose that $n + S(k) = m + S(k)$. Then, by the definition of addition, $S(n + k) = n + S(k) = m + S(k) = S(m + k)$. Since S is $1 - 1$ on $\mathbb{N}$ by number (5) of Peano's Axioms, it follows that $n + k = m + k$, from which it follows in turn that $n = m$ by the Induction Hypothesis.

Thus the Cancellation Law for addition on the natural numbers holds by induction, *i.e.* for all n, m, $k \in \mathbb{N}$, if $n + k = m + k$, then $n = m$. ■

Given that we have defined addition on the natural numbers, we can also define multiplication on the natural numbers by recursion as follows:

- For all $n \in \mathbb{N}$, $n \cdot 0 = 0$.
- For all $n \in \mathbb{N}$, if $n \cdot k$ has been defined, then $n \cdot S(k) = n \cdot k + n$.

Note that when evaluating the expression $n \cdot k + n$, we use the normal convention that unless operations are explicitly grouped otherwise, multiplications are performed before additions. Thus $n \cdot k + n$ means $(n \cdot k) + n$, not $n \cdot (k + n)$.

Addition and multiplication are connected a little more generally via the following rule:

DISTRIBUTIVE LAW (for $\cdot$ over $+$ on $\mathbb{N}$): For all a, b, $c \in \mathbb{N}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

2. Prove the Distributive Law for $\cdot$ over $+$ on $\mathbb{N}$. *[5]*

SOLUTION. We will proceed by induction on c.

Base Step. $[c = 0]$ For any and all $a,\ b \in \mathbb{N}$, using the definitions of addition and multiplication, we have $a \cdot (b + 0) = a \cdot b = a \cdot b + 0 = a \cdot b + b \cdot 0$, as required.

Inductive Hypothesis. Assume that for some $c \geq 0$ and all $a,\ b \in \mathbb{N}$, we have that $a \cdot (b + c) = a \cdot b + a \cdot c$.

Inductive Step. $[c \to S(c)]$ We use the definitions of addition and multiplication, plus the associative law for addition and the Inductive Hypothesis, to get what we need:

$$a \cdot (b + S(c)) = a \cdot S(b+c) = a \cdot (b+c) + a = (a \cdot b + a \cdot c) + c = a \cdot b + (a \cdot c + c) = a \cdot b + a \cdot S(c)$$

Thus the Distributive Law for $\cdot$ over $+$ on $\mathbb{N}$ is true by induction. ■