Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Solutions to Assignment #5
Going down!
Due on Friday, 15 October.

Recall from class that the natural numbers can be built up as follows from the empty set using the successor function, $S(x) = x \cup \{x\}$.

$$\begin{aligned}
&0 = \emptyset \\
&1 = S(0) = 0 \cup \{0\} = \{0\} \\
&2 = S(1) = 1 \cup \{1\} = \{0, 1\} \\
&3 = S(2) = 2 \cup \{2\} = \{0, 1, 2\} \\
&\vdots \\
&n = S(n-1) = (n-1) \cup \{n-1\} = \{0, 1, 2, \ldots, n-1\} \\
&\vdots
\end{aligned}$$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ then denotes the set of all natural numbers. (Of course, we had to add the Axiom of Infinity to our collection of axioms for set theory to ensure that $\mathbb{N}$ is indeed a set.) Note that each natural number, considered as a set, is just the set of all of its predecessors. One pleasant consequence of this is that we can now define $<$ on the natural numbers very easily:

$a < b$ for natural numbers a and b if and only if $a \in b$.

Recall also the Descending Chain Condition from Assignment #3, which we rephrase here a little to apply to the natural numbers in particular:

($\downarrow$) *(Descending Chain Condition)* Every strictly decreasing sequence of natural numbers is finite.

That is, if you have a sequence of natural numbers $a_0 > a_1 > a_2 > \cdots$, then it cannot be infinite.

1. Using the definitions of the natural numbers and $<$ given above, prove the Descending Chain Condition is true of the natural numbers. *[3]*

SOLUTIONS. In what follows, we'll take "descending chain" to mean "strictly decreasing sequence of natural numbers". It's so much shorter to write or type! :-)

Solution the first – straightforward and a bit informal. Suppose $a_0 > a_1 > a_2 > \cdots$ is a descending chain of natural numbers. Since $a_k < a_0$ means that $a_k \in a_0$ for each element a_k (with $k \geq 1$) of the descending chain, the descending chain (for $k \geq 1$) can have no more elements than $a_0 = \{0, 1, 2, \ldots, a_k - 1\}$. As a_0 is a finite set, and each element of the descending chain is different from the others – if $a_k < a_\ell$, then $a_k \in a_\ell$, so $a_k \neq a_\ell$ – this means that the descending chain must be finite as well. $\square$

Solution the second – using the Axiom of Foundation directly. [It was used indirectly in the first solution. Where?] Consider the elements of the descending sequence as the elements of a set: $A = \{a_0, a_1, a_1, \dots\}$. $A \neq \emptyset$ because we must have $a_0 \in A$ to even have the beginning of a sequence. It follows by the Axiom of Foundation that there is some $a_k \in A$ such that $a_k \cap A = \emptyset$. a_k must be the last and smallest element of the sequence: if there was a $k+1$st element of the sequence, then we would have $a_{k+1} < a_k$, so we would have $a_{k+1} \in a_k$ and $a_{k+1} \in A$, contradicting $a_k \cap A = \emptyset$. Thus the descending chain must terminate at a_k for some $k \geq 0$, and hence must be finite. $\square$

Solution the third – using the Peano Axioms for the natural numbers. As was noted in class, the natural numbers as defined above satisfy the Peano Axioms. Let $X \subseteq \mathbb{N}$ be defined as follows:

$$X = \{\, n \in \mathbb{N} \mid \text{Every descending chain starting from } a_0 = n \text{ is finite} \,\}$$

Observe that $0 \in X$ since if a descending chain has $a_0 = 0$, there is nowhere in $\mathbb{N}$ to descend to from 0, so $a_0 = 0$ is the entire chain. A one-element chain is certainly finite ...

Now suppose that $n \in X$ for some $n \geq 0$, *i.e.* that every descending chain starting from $a_0 = n$ is finite. Suppose we have a descending chain starting from $a_0 = S(n) = n+1$. If a_0 is the entire chain, the chain is finite. If $a_1 = n$, the the descending chain from a_1 must be finite since $n \in X$, and hence the descending chain from $a_0 = n+1$ is also finite, since it has only one more element. If $a_1 < n$, then the descending chain $n > a_1 > \cdots$ is finite since $n \in X$, but this chain is just as long as the chain $a_0 > a_1 > \cdots$, so the descending chain from a_0 is also finite. Thus, in each of the three possible cases, any descending chain from $a_0 = n+1$ must be finite, so $S(n) = n+1 \in X$.

By number (6) of the Peano Axioms, it follows that $X = \mathbb{N}$, so every descending chain starting from any natural number is finite, *i.e.* every descending chain of natural numbers is finite. $\square$

Solution the fourth – by induction. We will proceed by induction on n to show that every descending chain of natural numbers starting at n is finite. Unsurprisingly, this turns out to be very similar to the third solution above. There is a reason the Peano Axiom number (6) is often called the induction axiom, after all.

Base Step. ($n = 0$) If a descending chain has $a_0 = 0$, there is nowhere in $\mathbb{N}$ to descend to from 0, so $a_0 = 0$ must be the entire chain, and a one-element chain is certainly finite.

Inductive Hypothesis. Assume that every descending chain starting from $a_0 = n$ is finite.

Inductive Step. Suppose we have a descending chain starting from $a_0 = n+1$. If a_0 is the entire chain, the chain is finite. If $a_1 = n$, the the descending chain from a_1 must be finite by the inductive hypothesis, and hence the descending chain from $a_0 = n+1$ is also finite, since it has only one more element. If $a_1 < n$, then the descending chain $n > a_1 > \cdots$ is finite by the inductive hypothesis, but this chain is just as long as the chain $a_0 > a_1 > \cdots$, so the descending chain from a_0 is also finite. Thus, in each of the three possible cases, any descending chain from $a_0 = n+1$ must be finite.

It follows by induction that every descending chain starting from any natural number n is finite, *i.e.* every descending chain of natural numbers is finite. ■