# Mathematics 2200H – Mathematical Reasoning

Trent University, Fall 2021

## Solutions to Assignment #4
## "The Ramans do everything in threes."†

*Due on Friday, 8 October.*

Let's define a *set of threes* to be a set all of whose elements are sets which have exactly three elements. The object of this assignment is to devise a formula $\varphi$ with one free variable, say $x_0$, in a pretty minimalist language for set theory, such that $\varphi$ is true exactly when $x$ is a set of threes. (Recall that an occurrence of a variable in a formula is free if it is not in the scope of a quantifier in that formula.) Here is a formal definition of the first-order language for set theory $\varphi$ should be a formula of:

The symbols of the language are as follows:

*Variables:* $x_0$, $x_1$, $x_2$, ...
*Connectives:* $\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$
*Quantifiers:* $\forall$, $\exists$
*Parentheses:* (, )
*Equality:* $=$
*Set Membership:* $\in$ (a 2-place relation)
Just to be paranoid: all of the above symbols are distinct, none is a substring of any other, and there are no other symbols in the language.

Note that the only terms of this language are the variables, as there are no constant symbols (not even for the empty set) or function symbols.

The formulas of the language are defined as follows:

1. For any variables $x_i$ and $x_j$ of the language, $(x_i = x_j)$ and $(x_i \in x_j)$ are formulas of the language.
2. If $\varphi$ and $\psi$ are any formulas of the language, then $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \to \psi)$, and $(\varphi \leftrightarrow \psi)$ are also formulas of the language.
3. If $\varphi$ is any formula of the language and $x_i$ is any variable of the language, then $(\forall x_i \varphi)$ and $(\exists x_i \varphi)$ are also formulas of the language.
4. No string of symbols of the language is a formula of the language unless it was formed using (possibly many applications of) rules 1–3 above.

This language is inefficient in many ways, as it lacks most of the specialized symbols normally used in set theory, but at least it is uncomplicated as first-order languages go.

**1.** Give a formula $\psi$ in the language specified above that has $x_1$ as its only free variable and is true exactly when $x_1$ has exactly three elements. *[5]*

Solution. *Cut the first:* We want $x_1$ to have three elements, so let's try "there are $x_2$, $x_3$, and $x_4$ which are all in $x_1$":

$$\exists x_2 \exists x_3 \exists x_4 \, (x_2 \in x_1 \wedge x_3 \in x_1 \wedge x_4 \in x_1)$$

† From the end of Arthur C. Clarke's novel *Rendezvous with Rama*.

Unfortunately, this doesn't quite work, since this formula will be true whenever $x_1$ is any non-empty set because there is nothing in it that requires the elements $x_2$, $x_3$, and $x_4$ to be different from each other. We can fix this ...

*Cut the second:* We want to make sure that $x_1$ has three *different* elements, so let's try "there are $x_2$, $x_3$, and $x_4$ which are all in $x_1$ and are not equal to each other":

$$\exists x_2 \exists x_3 \exists x_4 \, (x_2 \in x_1 \land x_3 \in x_1 \land x_4 \in x_1 \land (\neg x_2 = x_3) \land (\neg x_2 = x_4) \land (\neg x_3 = x_4))$$

Unfortunately, this still doesn't quite work, since this formula will be true whenever $x_1$ has at least three different elements. If it had, say, four different elements, any three of them could fill in for $x_2$, $x_3$, and $x_4$. We can fix this, too ...

*Cut the third:* We also want to make sure that $x_1$ has *no more than* three different elements, so let's try "there are $x_2$, $x_3$, and $x_4$ which are all in $x_1$ and are not equal to each other, and any $x_5$ in $x_1$ is equal to $x_2$, $x_3$, or $x_4$":

$$\begin{gathered}\exists x_2 \exists x_3 \exists x_4 \, (x_2 \in x_1 \land x_3 \in x_1 \land x_4 \in x_1 \land (\neg x_2 = x_3) \land (\neg x_2 = x_4) \land (\neg x_3 = x_4) \\ \land \forall x_5 \, (x_5 \in x_1 \to (x_5 = x_2 \lor x_5 = x_3 \lor x_6 = x_4)))\end{gathered}$$

This works! It's not quite an official formula of the given language, though, since we've omitted various parentheses in the interests of readability. This too we can fix ...

*Cut the fourth:* Let's see – we need parentheses about all formulas, including equalities and uses of the element-of relation. Note also that our connectives, other than negation, are binary, so if we use them to glue together three or more formulas, we can only do it two at a time. Here goes:

$$\begin{aligned}&(\exists x_2 \, (\exists x_3 \, (\exists x_4 \, (((((x_2 \in x_1) \land (x_3 \in x_1)) \land (x_4 \in x_1)) \\ &\qquad \land (((\neg \, (x_2 = x_3)) \land (\neg \, (x_2 = x_4))) \land (\neg \, (x_3 = x_4)))) \\ &\qquad \land (\forall x_5 \, ((x_5 \in x_1) \to (((x_5 = x_2) \lor (x_5 = x_3)) \lor (x_6 = x_4))))))))\end{aligned}$$

This will hopefully do for $\psi$, even if we're sticklers for the language. In practice, the version in the third cut above would do for almost everyone. Whew! ■

**2.** Give a formula $\varphi$ in the language specified above that has $x_0$ as its only free variable and is true exactly when $x_0$ is a set of threes. *[5]*

SOLUTION. By definition, $x_0$ is a set of threes exactly when every element of $x_0$ has exactly three elements, so let's try "every $x_1$ in $x_0$ satisfies $\psi$", where $\psi$ is the formula we obtained in solving **1** above. (Recall that $\psi$ will be true precisely when $x_1$ has exactly three elements.) Let's try it:

$$\forall x_1 \, (x_1 \in x_0 \to \psi)$$

This is disappointingly uncomplicated – almost all of the work is buried in $\psi$ – so we make it look more impressive by putting in the omitted parentheses and writing out $\psi$:

$$\begin{aligned}&(\forall x_1 \, ((x_1 \in x_0) \to \\ &(\exists x_2 \, (\exists x_3 \, (\exists x_4 \, (((((x_2 \in x_1) \land (x_3 \in x_1)) \land (x_4 \in x_1)) \\ &\qquad \land (((\neg \, (x_2 = x_3)) \land (\neg \, (x_2 = x_4))) \land (\neg \, (x_3 = x_4)))) \\ &\qquad \land (\forall x_5 \, ((x_5 \in x_1) \to (((x_5 = x_2) \lor (x_5 = x_3)) \lor (x_6 = x_4)))))))) \\ &\qquad ))\end{aligned}$$

That's all, folks! ■