Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2021

Final Examination

Due on Friday, 17 December, on paper or via Blackboard.

Instructions: Do both of parts ¬ and →, and, if you wish, part ∨ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part ¬. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Define the logical connective $\uparrow$ via the following truth table:

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

a. Write a formula equivalent to $A \uparrow B$ using the logical connectives $\neg$ and $\rightarrow$. *[2]*
b. Write formulas equivalent to $\neg A$, $A \vee B$, $A \wedge B$, and $A \Rightarrow B$ using just the connective $\uparrow$. *[8]*

NOTE: In both **a** and **b** you may use the connective(s) you are supposed to use as many times as you like in the desired formula.

2. The Fibonacci sequence $\{f_n\}$ is defined inductively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Use induction to show that for all $n \geq 0$,

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] . \qquad [10]$$

3. Suppose A and B are schnitts and $C = \{\, a - b \mid a \in A \text{ and } b \in B \,\}$. Determine whether C can be a schnitt or not. *[10]*

4. Suppose $<_A$ is a (strict) linear order on a set A with the property that every non-empty subset $B \subseteq A$ has both a greatest element and a least element in the linear order $<_A$. Show that A must be a finite set. *[10]*

Part →. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Show that $\mathbb{Z}^2 = \{\, (a,b) \mid a,\, b \in \mathbb{Z} \,\}$ is countable. *[10]*

More problems on page 2!

6. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet nine inhabitants: Dave, Alice, Mel, Bob, Abe, Bart, Joe, Peggy and Rex. Dave says that both Rex is a knight and Abe is a knave. Alice says, "Neither Abe nor Bob are knaves." Mel claims that at least one of the following is true: that Bart is a knight or that Abe is a knave. Bob tells you that Peggy is a knave. Abe says that Dave is a knave. Bart claims that Peggy is a knave. Joe claims that neither Peggy nor Abe are knaves. Peggy claims that either Rex is a knight or Dave is a knight. Rex says that Peggy would tell you that Bart is a knight. Determine, as best you can, which of the nine are knights and which are knaves. *[10]*

7. **a.** Give an example of ordinals α, β, and $\gamma \neq 0$ such that $\alpha \neq \beta$ but $\alpha \cdot \gamma = \beta \cdot \gamma$. *[3]*
b. Prove the left cancellation law for ordinal addition: for any ordinals α, β, and γ, if $\gamma + \alpha = \gamma + \beta$, then $\alpha = \beta$. *[7]*

8. *Pentominoes* are shapes obtained by gluing five 1×1 squares together full edge to full edge. Two pentominoes that can be made congruent via reflections (*i.e.* flips) or rotations are considered to be the same. Find all twelve pentominoes and an arrangement of all of them into a 6×10 rectangle. *[10]*

9. A *perfect number* is a natural number $n > 0$ which is the sum of its positive divisors other than itself. For example, 6 is a perfect number because $6 = 1 + 2 + 3$. Suppose $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime numbers. Show that n is a perfect number. *[10]*

10. Define a binary relation W on the real numbers by $r \mathrm{W} s \iff r = s + 2\pi k$ for some integer k. The *wheel numbers* are the equivalence classes of this relation, *i.e.* $\mathbb{W} = \{ [r]_\mathrm{W} \mid r \in \mathbb{R} \}$. Define addition for wheel numbers by $[r]_\mathrm{W} + [s]_\mathrm{W} = [r+s]_\mathrm{W}$, and let $0_\mathrm{W} = [0]_\mathrm{W}$ and $1_\mathrm{W} = [1]_\mathrm{W}$.

- **a.** Show that addition for the wheel numbers is well-defined. *[2]*
- **b.** Show that addition for the wheel numbers is commutative and associative. *[4]*
- **c.** Can one define a linear order $<_\mathrm{W}$ on $\mathbb{W}$ such that $w <_\mathrm{W} w + 1_\mathrm{W}$ for all $w \in \mathbb{W}$? Either give a definition of such a linear order with an explanation of why it works, or explain why there cannot be such a linear order. *[4]*
- **d.** *(Bonus!)* Why did your instructor call them the wheel numbers? *[0.5]*

11. Recall that a cardinal number is an ordinal whose cardinality is strictly greater than that of each lesser ordinal. Suppose κ_n, for $n \in \mathbb{N}$, is a strictly increasing sequence of cardinal numbers and that $\kappa = \bigcup \{ \kappa_n \mid n \in \mathbb{N} \}$. Show that κ is also a cardinal number. *[10]*

[Total = 80]

Part V. Bonus!

Λ. Write an original poem about logic or mathematics. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. ENJOY THE BREAK EVEN MORE!